Image Anal Stereol 2025;44:131-145 doi: 10.5566/ias.3603
Original Research Paper

RECURRENCE RELATIONS FOR JACOBI WEIGHTED ORTHOGONAL POLYNOMIALS ON THE TRIANGULAR DOMAIN

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(Received March 31, 2025; revised June 28, 2025; accepted June 28, 2025)

ABSTRACT

In this paper, we present recurrence relations for the Jacobi weighted orthogonal polynomials $P_{n,r}^{(\alpha,\beta,\gamma)}(u,v,w)$ with $r = 0,1,\ldots,n$, where $n \geq 0$, defined on the triangular domain $T = \{(u,v,w) : u,v,w \geq 0, u+v+w = 1\}$ for values of $\alpha,\beta,\gamma > -1$. In particular, we construct univariate recurrence relations for Jacobi polynomials when $w = 0$, considering three specific cases. These recurrence relations provide an efficient and straightforward alternative for computing Jacobi polynomials, offering a simpler approach compared to traditional methods.

Keywords: Bernstein polynomials, Jacobi polynomials, orthogonal polynomials, recurrence relations, triangular domains.

INTRODUCTION

In recent years, there has been growing interest in developing analytical and numerical approaches for solving various classes of differential and integral equations arising in applied mathematics, physics, and engineering. This trend is reflected in a range of studies addressing the stability of nonlinear systems (Hajaj *et al.*, 2025), generalized matrix functions such as the Mittag-Leffler function (Batiha *et al.*, 2024), and spectral properties of structured matrices like the Frobenius companion matrix (Batiha *et al.*, 2023). Additional works have explored mapping properties of integral operators (Hawawsheh *et al.*, 2023), phase change problems such as the Stefan problem (Merabti *et al.*, 2023), and inequalities relevant to numerical integration (Alshanti *et al.*, 2023). Furthermore, classical computational techniques continue to be refined (Batiha, 2011), while modern research has also emphasized the formulation of efficient algorithms for solving Volterra integro-differential equations (Anakira *et al.*, 2023) and fractional differential equations using optimized decomposition strategies (Farraj *et al.*, 2023). Recent developments also include the study of fractional dynamical systems under uncertainty (Berir, 2024) and functional analytic frameworks involving fractional operators in abstract metric settings (Qawaqneh, 2024).

Polynomials play a fundamental role in various areas of mathematics, including computational mathematics, approximation theory, and numerical analysis. Orthogonal polynomials, particularly Jacobi polynomials, are extensively used due to their relevance in solving differential equations, polynomial approximation, and modeling physical phenomena. The study of two-variable orthogonal polynomials has a long and rich history, with foundational work by Koornwinder (Koornwinder, 1975) providing essential insights into multivariate analogues of classical systems. Further comprehensive treatments can be found in Dunkl and Xu (Dunkl *et al.*, 2014) and in the monograph by Koekoek et al. (Koekoek *et al.*, 2010). This paper builds on such foundational contributions by focusing on Jacobi-weighted orthogonal polynomials on triangular domains.

This study focuses on Jacobi polynomials, both univariate and bivariate, defined on the triangular domain. These polynomials satisfy specific orthogonality relations and provide a solid foundation for applications in applied mathematics and physics. A key aspect of Jacobi polynomials is their recurrence relations, which offer an efficient and systematic method for generating higher-degree polynomials from lower-degree ones.

The primary objective of this work is to derive and analyze recurrence relations for Jacobi weighted orthogonal polynomials $P_{n,r}^{(\alpha,\beta,\gamma)}(u,v,w)$ on the triangular domain $T = \{(u,v,w) : u,v,w \geq 0, u+v+w = 1\}$, where $\alpha,\beta,\gamma > -1$. Special attention is given to the case when $w = 0$, where these polynomials reduce to univariate Jacobi polynomials. The recurrence relations established in this study provide a more efficient alternative to standard computational methods, facilitating accurate and rapid computation of Jacobi polynomials.

The results presented in this paper contribute to advancing polynomial approximation techniques, offering significant improvements in computational efficiency and accuracy. These findings have broad implications for applications in numerical analysis, computational mathematics, and related fields.

FUNDAMENTAL CONCEPTS

This section introduces the fundamental concepts necessary to derive and analyze the recurrence relations of Jacobi weighted orthogonal polynomials on the triangular domain.

JACOBI POLYNOMIALS ON THE TRIANGULAR DOMAIN

Jacobi polynomials, denoted by $P_{n,r}^{(\alpha,\beta,\gamma)}(u,v,w)$, are defined on the triangular domain

$$T = \{(u,v,w) : u,v,w \geq 0, u+v+w = 1\}$$

where $\alpha,\beta,\gamma > -1$. These polynomials are orthogonal with respect to the Jacobi weight function

$$w^{(\alpha,\beta,\gamma)}(u,v,w) = u^{\alpha}v^{\beta}(1-w)^{\gamma}.$$

The recurrence relations for these polynomials are essential for generating higher-degree polynomials from lower-degree ones, thereby improving computational efficiency and enabling the approximation of complex functions with greater accuracy.

UNIVARIATE AND BIVARIATE CASES

For the univariate case, when $w = 0$, Jacobi polynomials reduce to the classical univariate Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, which are defined on the interval $[-1,1]$ and satisfy orthogonality with respect to the weight function

$$\rho(x) = (1-x)^{\alpha}(1+x)^{\beta}.$$

In the bivariate case, Jacobi polynomials are expressed in terms of Bernstein basis polynomials, allowing for a compact representation and efficient computation. The explicit formulation and recurrence relations for these polynomials are derived in subsequent sections.

BERNSTEIN REPRESENTATION OF JACOBI POLYNOMIALS

Jacobi polynomials on the triangular domain can be expressed in Bernstein basis form, which facilitates efficient computation and representation. This representation allows the derivation of recurrence relations that simplify the computation of higher-degree polynomials.

TRIANGULAR DOMAIN AND BARYCENTRIC COORDINATES

The triangular domain $T = \{(u,v,w) : u,v,w \geq 0, u+v+w = 1\}$ can be interpreted using barycentric coordinates, where any point P inside the triangle formed by the vertices P_1, P_2, and P_3 can be expressed as:

$$P = uP_1 + vP_2 + wP_3 \quad \text{with } u+v+w = 1.$$

This coordinate system is useful for expressing bivariate Jacobi polynomials on the triangular domain.

ORTHOGONAL POLYNOMIALS

Orthogonal polynomials play a central role in various branches of mathematics, including approximation theory, numerical analysis, and mathematical physics. They provide a systematic framework for approximating functions and solving differential equations efficiently. In this section, we define orthogonal polynomials and highlight their significance in mathematical theory and applications.

Definition 1 *A sequence of polynomials $\{P_n(x)\}_{n=0}^{\infty}$ is said to be* ***orthogonal*** *with respect to a weight function $\rho(x)$ on an interval $[a,b]$ if*

$$\langle P_i, P_j \rangle = \int_a^b P_i(x)P_j(x)\rho(x)\,dx = 0, \quad \text{for } i \neq j.$$

If $i = j$, the integral becomes

$$\int_a^b [P_i(x)]^2 \rho(x)\,dx \geq 0.$$

When this integral equals 1, the polynomials are said to be ***orthonormal****. For a more detailed discussion, see (Dunkl et al., 2014).*

ILLUSTRATIVE EXAMPLES OF ORTHOGONAL POLYNOMIALS

Example 1 *Chebyshev polynomials are a prominent example of orthogonal polynomials on the interval* $[-1,1]$*, with respect to the weight function*

$$\rho(x) = \frac{1}{\sqrt{1-x^2}},$$

which is known as the Chebyshev weight function of the first kind. The Chebyshev polynomial of degree n is defined by the formula:

$$T_n(x) = \cos(n\cos^{-1}(x)). \tag{1}$$

It is easy to verify that:

$$T_0(x) = \cos(0\cdot\cos^{-1}(x)) = \cos(0) = 1,$$
$$T_1(x) = \cos(\cos^{-1}(x)) = x.$$

To derive the Chebyshev orthogonal polynomials for $n \geq 2$*, we substitute* $\theta = \cos^{-1}(x)$ *in Eq. (1), giving:*

$$T_n(\theta) = \cos(n\theta).$$

By applying standard trigonometric identities, we obtain the recurrence relation for Chebyshev polynomials:

$$T_{n+1}(\theta) = \cos((n+1)\theta) = \cos n\theta\cos\theta - \sin n\theta\sin\theta,$$
$$T_{n-1}(\theta) = \cos((n-1)\theta) = \cos n\theta\cos\theta + \sin n\theta\sin\theta.$$

Adding these two equations gives:

$$T_{n+1}(\theta) + T_{n-1}(\theta) = 2\cos n\theta\cos\theta.$$

Since $\theta = \cos^{-1}(x)$*, we have* $x = \cos\theta$ *and* $\cos\theta = T_n(\theta)$*. Substituting these values, we obtain the standard recurrence relation for Chebyshev polynomials:*

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x), \quad n = 2,3,4,\ldots,$$

with initial conditions:

$$T_0(x) = 1, \quad T_1(x) = x.$$

Example 2 *Hermite polynomials are defined in (Szegő, 1975) by the orthogonality condition:*

$$\int_{-\infty}^{\infty} e^{-x^2} H_n(x)H_m(x)\,dx = \sqrt{\pi}\,n!\,\delta_{nm},$$

for $n,m = 0,1,2,\ldots$*, where* δ_{nm} *is the Kronecker delta. The coefficient of* x^n *in the n-th Hermite polynomial is positive.*

The Hermite polynomials satisfy the following recurrence relation:

$$H_n(x) = 2xH_{n-1}(x) - 2(n-1)H_{n-2}(x),$$

$n = 2,3,4,\ldots$*, with the initial conditions:*

$$H_0(x) = 1, \quad H_1(x) = 2x.$$

Example 3 *Legendre polynomials are orthogonal on the interval* $[-1,1]$ *with respect to the weight function:*

$$\rho(x) = 1.$$

The recursive relation is used to generate Legendre polynomials for $n \geq 2$*. These polynomials can be constructed using the Gram-Schmidt process.*

Theorem 1 *The set of polynomial functions* $\{P_0(x), P_1(x), \ldots, P_n(x)\}$ *defined in the following way is orthogonal on* $[a,b]$ *with respect to the weight function* $\rho(x)$*. The first two polynomials are given by:*

$$P_0(x) \equiv 1, \quad P_1(x) = x - B_1,$$

where

$$B_1 = \frac{\int_a^b x\rho(x)[P_0(x)]^2\,dx}{\int_a^b \rho(x)[P_0(x)]^2\,dx}.$$

For $k \geq 2$*, the recurrence relation is:*

$$P_k(x) = (x - B_k)P_{k-1}(x) - C_kP_{k-2}(x), \quad \forall x \in [a,b],$$

where

$$B_k = \frac{\int_a^b x\rho(x)[P_{k-1}(x)]^2\,dx}{\int_a^b \rho(x)[P_{k-1}(x)]^2\,dx},$$
$$C_k = \frac{\int_a^b x\rho(x)P_{k-1}(x)P_{k-2}(x)\,dx}{\int_a^b \rho(x)[P_{k-2}(x)]^2\,dx}.$$

To compute Legendre polynomials, we substitute $\rho(x) = 1$ *and consider the interval* $[-1,1]$*. To find* $P_2(x)$*, we compute* B_1, B_2*, and* C_2 *as follows:*

$$B_1 = \frac{\int_{-1}^1 x\,dx}{\int_{-1}^1 dx} = 0, \quad B_2 = \frac{\int_{-1}^1 x^3\,dx}{\int_{-1}^1 x^2\,dx} = 0,$$
$$C_2 = \frac{\int_{-1}^1 x^2\,dx}{\int_{-1}^1 dx} = \frac{1}{3}.$$

The first four Legendre polynomials, computed using these formulas, are:

$$P_0(x) = 1,$$
$$P_1(x) = x,$$
$$P_2(x) = x^2 - \frac{1}{3},$$
$$P_3(x) = x^3 - \frac{3}{5}x,$$
$$P_4(x) = x^4 - \frac{6}{7}x^2 + \frac{3}{35}.$$

JACOBI POLYNOMIALS

To provide a clear foundation for the development of recurrence relations on triangular domains, we begin with the classical theory of univariate Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$. This allows us to build a natural and logical extension to the bivariate case.

UNIVARIATE JACOBI POLYNOMIALS

Univariate Jacobi polynomials, denoted by $P_n^{(\alpha,\beta)}(x)$, are orthogonal polynomials of degree $n \geq 0$ with respect to the weight function:

$$\rho(x) = (1-x)^{\alpha}(1+x)^{\beta}, \quad \alpha, \beta > -1,$$

defined on the interval $[-1,1]$. These polynomials can also be computed for $n \geq 2$ using the general recurrence formula:

$$\begin{aligned} P_n^{(\alpha,\beta)}(x) &= (2n+\alpha+\beta-1) \\ &\times \frac{\left[(2n+\alpha+\beta)(2n+\alpha+\beta-2)x+\alpha^2-\beta^2\right]}{2n(n+\alpha+\beta)(2n+\alpha+\beta-2)} P_{n-1}^{(\alpha,\beta)} \\ &\times -\frac{2(n+\alpha-1)(n+\beta-1)(2n+\alpha+\beta)}{2n(n+\alpha+\beta)(2n+\alpha+\beta-2)} P_{n-2}^{(\alpha,\beta)}(x), \end{aligned} \tag{2}$$

where the initial polynomials are given by:

$$P_0^{(\alpha,\beta)}(x) = 1, \; P_1^{(\alpha,\beta)}(x) = \frac{1}{2}(\alpha+\beta+2)x + \frac{1}{2}(\alpha-\beta).$$

Jacobi polynomials of degree r, denoted by $P_r^{(\alpha,\beta)}(x)$ for $x \in [0,1]$, can also be expressed using the Bernstein representation, see (Rababah, 2004; Rababah , 2003):

$$P_r^{(\alpha,\beta)}(x) = \sum_{i=0}^{r} (-1)^{r-i} \frac{\binom{r+\alpha}{i}\binom{r+\beta}{r-i}}{\binom{r}{i}} b_i^r(x), \tag{3}$$

for $r = 0,1,\ldots$. Based on the univariate Jacobi framework, we now present the bivariate Jacobi polynomials defined over the triangular domain using barycentric coordinates and Bernstein representation.

BIVARIATE JACOBI POLYNOMIALS

Bivariate Jacobi polynomials, denoted by $P_{n,r}^{(\alpha,\beta,\gamma)}(u,v,w)$, are polynomials of degree n with respect to the Jacobi weight function:

$$w^{(\alpha,\beta,\gamma)}(u,v,w) = u^{\alpha}v^{\beta}(1-w)^{\gamma},$$

where $\alpha,\beta,\gamma > -1$ on the triangular domain:

$$T = \{(u,v,w) : u,v,w \geq 0, u+v+w = 1\}.$$

These polynomials are expressed in Bernstein basis form as:

$$\begin{aligned} P_{n,r}^{(\alpha,\beta,\gamma)}(u,v,w) &= \sum_{i=0}^{r} (-1)^{r-i} \frac{\binom{r+\alpha}{i}\binom{r+\beta}{r-i}}{\binom{r}{i}} b_i^r(u,v) \\ &\times \sum_{j=0}^{n-r} (-1)^j \binom{n+r+\sigma+1}{j} b_j^{n-r}(w,u+v), \end{aligned} \tag{4}$$

where $\sigma = \alpha+\beta+\gamma$, and $r = 0,1,\ldots,n$, with $n \geq 0$.

These polynomials, $P_{n,r}^{(\alpha,\beta,\gamma)}(u,v,w)$, satisfy the following properties, as outlined in the theorems below.

Theorem 2 *For all $\alpha,\beta,\gamma > -1$, the bivariate Jacobi polynomials $P_{n,r}^{(\alpha,\beta,\gamma)}(u,v,w)$ of degree $n \geq 1$ satisfy:*

$$P_{n,r}^{(\alpha,\beta,\gamma)}(u,v,w) \in \mathscr{L}_n, \quad r = 0,1,\ldots,n,$$

where

$$\mathscr{L}_n = \{P \in \Pi_n : P \perp \Pi_{n-1}\}.$$

***Proof* 1** *See Theorem 3 in (Rababah, 2005).*

Theorem 3 *For $r \neq s$ and all $\alpha,\beta,\gamma > -1$, the bivariate Jacobi polynomials $P_{n,r}^{(\alpha,\beta,\gamma)}(u,v,w)$ and $P_{n,s}^{(\alpha,\beta,\gamma)}(u,v,w)$ are orthogonal with respect to the weight function:*

$$w^{(\alpha,\beta,\gamma)}(u,v,w) = u^{\alpha}v^{\beta}(1-w)^{\gamma}.$$

***Proof* 2** *For $r \neq s$, let*

$$P_{n,r}^{(\alpha,\beta,\gamma)}(u,v,w) = P_r^{(\alpha,\beta)}\left(\frac{u}{1-w}\right)(1-w)^r q_{n,r}(w),$$

and

$$P_{n,s}^{(\alpha,\beta,\gamma)}(u,v,w) = P_s^{(\alpha,\beta)}\left(\frac{u}{1-w}\right)(1-w)^s q_{n,s}(w).$$

We want to show that the following integral equals zero:

$$\begin{aligned} I &= \int\int_T P_{n,r}^{(\alpha,\beta,\gamma)}(u,v,w) P_{n,s}^{(\alpha,\beta,\gamma)}(u,v,w) \\ &\quad \times w^{(\alpha,\beta,\gamma)}(u,v,w)\, dA, \\ &= \Delta \int_0^1 \int_0^{1-w} P_r^{(\alpha,\beta)}\left(\frac{u}{1-w}\right) \\ &\quad \times P_s^{(\alpha,\beta)}\left(\frac{u}{1-w}\right)(1-w)^{r+s} q_{n,r}(w) q_{n,s}(w) \\ &\quad \times w^{(\alpha,\beta,\gamma)}(u,v,w)\, du\, dw. \end{aligned}$$

Substitute $t = \frac{u}{1-w}$, which gives:

$$w^{(\alpha,\beta,\gamma)}(u,v,w) = t^{\alpha}(1-t)^{\beta}(1-w)^{\alpha+\beta+\gamma},$$

and since $du = dt(1-w)$, the integral becomes:

$$I = \Delta \int_0^1 \int_0^1 P_r^{(\alpha,\beta)}(t) P_s^{(\alpha,\beta)}(t)(1-w)^{r+s} q_{n,r}(w) \times q_{n,s}(w) t^{\alpha}(1-t)^{\beta}(1-w)^{\alpha+\beta+\gamma} dt(1-w)dw.$$

This can be separated into two integrals:

$$I = \Delta \int_0^1 P_r^{(\alpha,\beta)}(t) P_s^{(\alpha,\beta)}(t) t^{\alpha}(1-t)^{\beta} dt \times \int_0^1 q_{n,r}(w) q_{n,s}(w)(1-w)^{\alpha+\beta+\gamma+r+s+1} dw.$$

By the orthogonality property of Jacobi polynomials, the first integral equals zero for $r \neq s$, hence:

$$I = 0.$$

This completes the proof.

The bivariate polynomials $P_{n,r}^{(\alpha,\beta,\gamma)}(u,v,w)$, for $n \geq 0$ and $r = 0,1,\ldots,n$, form an orthogonal system over the triangular domain T with respect to the Jacobi weighted function. These polynomials can be represented in a triangular table as follows:

$$\begin{array}{llll} P_{0,0}^{(\alpha,\beta,\gamma)}(u,v,w) & & & \\ P_{1,0}^{(\alpha,\beta,\gamma)}(u,v,w) & P_{1,1}^{(\alpha,\beta,\gamma)}(u,v,w) & & \\ P_{2,0}^{(\alpha,\beta,\gamma)}(u,v,w) & P_{2,1}^{(\alpha,\beta,\gamma)}(u,v,w) & P_{2,2}^{(\alpha,\beta,\gamma)}(u,v,w) & \cdots \\ P_{n,0}^{(\alpha,\beta,\gamma)}(u,v,w) & P_{n,1}^{(\alpha,\beta,\gamma)}(u,v,w) & \ldots & P_{n,n}^{(\alpha,\beta,\gamma)}(u,v,w) \end{array} \tag{5}$$

The J-th row contains $J+1$ polynomials, each of degree J, where $J = 0,1,2,\ldots,n$.

If n is replaced by $n+t$ for any integer $t \geq 1$, the table is extended by calculating the polynomials in these additional rows using the recurrence formula (7):

$$\begin{array}{l} P_{n+1,0}^{(\alpha,\beta,\gamma)}(u,v,w), P_{n+1,1}^{(\alpha,\beta,\gamma)}(u,v,w), \ldots, P_{n+1,n+1}^{(\alpha,\beta,\gamma)}(u,v,w) \\ P_{n+2,0}^{(\alpha,\beta,\gamma)}(u,v,w),\ \ldots,\ P_{n+2,n+1}^{(\alpha,\beta,\gamma)}(u,v,w),\ P_{n+2,n+2}^{(\alpha,\beta,\gamma)}(u,v,w) \\ \vdots \\ P_{n+t,0}^{(\alpha,\beta,\gamma)}(u,v,w), \ldots, P_{n+t,n+t-1}^{(\alpha,\beta,\gamma)}(u,v,w), P_{n+t,n+t}^{(\alpha,\beta,\gamma)}(u,v,w) \end{array} \tag{6}$$

The aim of this study is to derive the recurrence relation for the system $\left\{P_{n,r}^{(\alpha,\beta,\gamma)}(u,v,w)\right\}_{r=0}^{n}$ where $\alpha,\beta,\gamma > -1$, and to establish the recurrence relation for this system on the plane where $\alpha+\beta+\gamma = 0$. This is achieved by considering three cases: when $r = n$, $r = n-1$, and $r \leq n-2$. Additionally, we derive a univariate recurrence relation for Jacobi polynomials when $w = 0$.

Remark 1 *The condition $\alpha+\beta+\gamma = 0$ is introduced here to facilitate algebraic simplification and reduce the computational burden associated with the general recurrence structure. This constraint helps isolate the essential behavior of the recurrence relations while maintaining the orthogonality properties of the Jacobi-weighted basis. A more general treatment with arbitrary values of $\alpha,\beta,\gamma > -1$ will be considered in future work to broaden the applicability of the proposed method.*

In the final section, we illustrate Jacobi polynomials in the cubic case on the triangular domain T, using various values of α, β, and γ. The Jacobi weighted orthogonal polynomials $\left\{P_{n,r}^{(\alpha,\beta,\gamma)}(u,v,w)\right\}_{r=0}^{n}$ are represented in a triangular table, where:

- The first row contains one polynomial of degree zero.
- The second row contains two polynomials of degree one.
- The $(n+1)$-th row contains $n+1$ polynomials, each of degree n.

These polynomials are presented in a triangular table in Section 6. Consequently, there are $\frac{1}{2}(n+1)(n+2)$ polynomials of degree $\leq n$. Each polynomial can be computed explicitly using formula (7).

However, using this formula is computationally expensive, requiring numerous operations and involving the multiplication of two summations. Therefore, a more efficient method for computing these polynomials is needed. In this study, we construct recurrence relations for the Jacobi weighted orthogonal system to provide a simpler and faster approach for populating the triangular table. In particular, we derive a recurrence relation under the constraint $\alpha+\beta+\gamma = 0$, where $\alpha,\beta,\gamma > -1$ and $w = 0$.

GENERAL RECURRENCE RELATION

This section introduces the recurrence relations for

$$P_{n,r}^{(\alpha,\beta,\gamma)}(u,v,w)$$

under the condition $\alpha+\beta+\gamma=0$. We begin with the basic recurrence relation and then explore specific cases.

- In the first case, where $r=n$, the recurrence relation is derived from the properties of $P_n^{(\alpha,\beta)}$.
- The second case considers $r=n-1$, where the recurrence relation is derived from the general formula.
- The final case deals with the scenario when $r\leq n-2$, providing a recurrence relation applicable to all such values of r.

Finally, we present the general recurrence relation that encompasses all of these cases.

In general, as shown in (Rababah, 2004; 2005), we have:

$$P_{n,r}^{(\alpha,\beta,\gamma)}(u,v,w)=P_r^{(\alpha,\beta)}\left(\frac{u}{1-w}\right)(1-w)^r q_{n,r,\sigma}(w), \tag{7}$$

for $r=0,1,2,\ldots,n$, where $\sigma=\alpha+\beta+\gamma$.

From equation (5.6) in (Rababah and Alqudah, 2005), we know:

$$q_{n,r}(w)=KP_{n-r}^{(0,2r+1)}(1-2w),$$
$$\sum_{j=0}^{n-r}(-1)^j\binom{n+r+1}{j}\binom{n-r}{j}w^j(1-w)^{n-r-j}$$
$$=K\sum_{j=0}^{n-r}(-1)^{n-r-j}.$$

In the same way, we have:

$$\sum_{j=0}^{n-r}(-1)^j\binom{n+r+\sigma+1}{j}\binom{n-r}{j}w^j(1-w)^{n-r-j}$$
$$=K^*\sum_{j=0}^{n-r}(-1)^{n-r-j}\binom{n+r+\sigma+1}{n-r-j}\binom{n-r}{j}$$
$$\times(1-2w)^j(2w)^{n-r-j}.$$

Here, $P_r^{(\alpha,\beta)}\left(\frac{u}{1-w}\right)$ is the Jacobi polynomial of degree r, and $q_{n,r,\sigma}(w)$ is a scalar multiple of

$$P_{n-r}^{(0,2r+\sigma+1)}(1-2w),$$

where:

$$q_{n,r,\sigma}(w)=K^*P_{n-r}^{(0,2r+\sigma+1)}(1-2w), \tag{8}$$

and $\sigma=\alpha+\beta+\gamma$, with K^* being a constant.

To derive the recurrence relation for the system $P_{n,r}^{(\alpha,\beta,\gamma)}(u,v,w)$, where $r=0,1,2,\ldots,n$, we consider the following three cases:

- Case 1: $r=n$, where the recurrence relation is derived from the properties of the univariate Jacobi polynomial.
- Case 2: $r=n-1$, where the recurrence relation is obtained using the general formula.
- Case 3: $r\leq n-2$, which provides a recurrence relation that holds for all such values of r.

These cases allow for a comprehensive derivation of the general recurrence relation for the system.

FIRST CASE $(r=n)$

In this case, we start from Eq. (8) with $r=n$. Now, we have

$$\begin{aligned} q_{n,n,\sigma}(w) &= K^*P_{n-n}^{(0,2n+\sigma+1)}(1-2w)\\ &= K^*P_0^{(0,2n+\sigma+1)}(1-2w)\\ &= K^*. \end{aligned} \tag{9}$$

The recurrence relation for the univariate Jacobi polynomial is given by:

$$P_n^{(\alpha,\beta)}\left(\frac{u}{1-w}\right)=\left[A_n\left(\frac{u}{1-w}\right)+B_n\right]\times P_{n-1}^{(\alpha,\beta)}\left(\frac{u}{1-w}\right)-C_nP_{n-2}^{(\alpha,\beta)}\left(\frac{u}{1-w}\right). \tag{10}$$

Substituting the above equations into Eq. (8) gives:

$$P_{n,n}^{(\alpha,\beta,\gamma)}(u,v,w)=\left[\left(A_n\left(\frac{u}{1-w}\right)+B_n\right)\times P_{n-1}^{(\alpha,\beta)}\left(\frac{u}{1-w}\right)-C_nP_{n-2}^{(\alpha,\beta)}\left(\frac{u}{1-w}\right)\right]\times(1-w)^nK^*. \tag{11}$$

From equations (8) and (9), we have:

$$P_{n-1,n-1}^{(\alpha,\beta,\gamma)}(u,v,w)=K^*P_{n-1}^{(\alpha,\beta)}\left(\frac{u}{1-w}\right)(1-w)^{n-1}, \tag{12}$$

and

$$P_{n-2,n-2}^{(\alpha,\beta,\gamma)}(u,v,w)=K^*P_{n-2}^{(\alpha,\beta)}\left(\frac{u}{1-w}\right)(1-w)^{n-2}. \tag{13}$$

Substitute equations (13) and (14) into equation (12) to obtain the recurrence relation for $P_{n,n}^{(\alpha,\beta,\gamma)}(u,v,w)$:

$$P_{n,n}^{(\alpha,\beta,\gamma)}(u,v,w) = [A_n(\frac{u}{1-w}) + B_n](1-w) \times P_{n-1,n-1}^{(\alpha,\beta,\gamma)}(u,v,w) - C_n(1-w)^2 P_{n-2,n-2}^{(\alpha,\beta,\gamma)}(u,v,w), \tag{14}$$

where A_n, B_n, and C_n are constants.

SECOND CASE ($r = n-1$)

Substituting $r = n-1$ into equation (8) gives:

$$P_{n,n-1}^{(\alpha,\beta,\gamma)}(u,v,w) = P_{n-1}^{(\alpha,\beta)}\left(\frac{u}{1-w}\right)(1-w)^{n-1} \times q_{n,n-1,\sigma}(w), \tag{15}$$

where $P_{n-1}^{(\alpha,\beta)}\left(\frac{u}{1-w}\right)$ is an orthogonal polynomial, and $q_{n,n-1,\sigma}(w)$ is a scalar multiple of $P_{n-(n-1)}^{(0,2(n-1)+\sigma+1)}(1-2w)$, i.e.,

$$q_{n,n-1,\sigma}(w) = K^* P_{n-(n-1)}^{(0,2(n-1)+\sigma+1)}(1-2w), \tag{16}$$

where K^* is a constant. From (Szegő, 1975), we know that

$$P_1^{(\alpha,\beta)}(x) = \frac{1}{2}(\alpha+\beta+2)x + \frac{1}{2}(\alpha-\beta).$$

Thus, we have

$$P_1^{(0,2(n-1)+\sigma+1)}(1-2w) = \frac{1}{2}(2n+\sigma+1)(1-2w) - \frac{1}{2}(2n+\sigma-1). \tag{17}$$

For simplicity, define

$$C^* = \frac{1}{2}(2n+\sigma+1), \quad D^* = \frac{1}{2}(2n+\sigma-1).$$

We now use the recurrence relation for $P_{n-1}^{(\alpha,\beta)}\left(\frac{u}{1-w}\right)$:

$$P_{n-1}^{(\alpha,\beta)}\left(\frac{u}{1-w}\right) = \left[A_{n-1}\left(\frac{u}{1-w}\right) + B_{n-1}\right] P_{n-2}^{(\alpha,\beta)}\left(\frac{u}{1-w}\right) - C_{n-1} P_{n-3}^{(\alpha,\beta)}\left(\frac{u}{1-w}\right). \tag{18}$$

Substitute these into equation (15) to obtain:

$$P_{n,n-1}^{(\alpha,\beta,\gamma)}(u,v,w) = \left[\left[A_{n-1}\left(\frac{u}{1-w}\right) + B_{n-1}\right] \times P_{n-2}^{(\alpha,\beta)}\left(\frac{u}{1-w}\right) - C_{n-1} P_{n-3}^{(\alpha,\beta)}\left(\frac{u}{1-w}\right)\right] \times (1-w)^{n-1} K^*(C^*(1-2w) - D^*). \tag{19}$$

Distributing the terms and associating the result to establish the Jacobi polynomials of degree $n-2$ and $n-3$, we get:

$$P_{n,n-1}^{(\alpha,\beta,\gamma)}(u,v,w) = \left[A_{n-1}\left(\frac{u}{1-w}\right) + B_{n-1}\right] \times P_{n-2}^{(\alpha,\beta)}\left(\frac{u}{1-w}\right)(1-w)^{n-2} \times K^*[(1-w)(C^*(1-2w) - D^*)] - C_{n-1} P_{n-3}^{(\alpha,\beta)}\left(\frac{u}{1-w}\right)(1-w)^{n-3} K^*[(1-w)^2(C^*(1-2w) - D^*)]. \tag{20}$$

Substitute the following equations:

$$P_{n-2,n-2}^{(\alpha,\beta,\gamma)}(u,v,w) = P_{n-2}^{(\alpha,\beta)}\left(\frac{u}{1-w}\right)(1-w)^{n-2} K^*,$$

and

$$P_{n-3,n-3}^{(\alpha,\beta,\gamma)}(u,v,w) = P_{n-3}^{(\alpha,\beta)}\left(\frac{u}{1-w}\right)(1-w)^{n-3} K^*.$$

Substitute these expressions into the previous equation to obtain:

$$P_{n,n-1}^{(\alpha,\beta,\gamma)}(u,v,w) = \left[A_{n-1}\left(\frac{u}{1-w}\right) + B_{n-1}\right] \times P_{n-2,n-2}^{(\alpha,\beta,\gamma)}(u,v,w)[(1-w)(C^*(1-2w) - D^*)] - C_{n-1} P_{n-3,n-3}^{(\alpha,\beta,\gamma)}(u,v,w)[(1-w)^2(C^*(1-2w) - D^*)] \tag{21}$$

where A_{n-1}, B_{n-1}, C_{n-1}, C^*, and D^* are constants.

THIRD CASE ($r \leq n-2$)

Herein, we have

for $r = 0, 1, 2, \ldots, n-2$. In this case, we express $q_{n,r,\sigma}(w)$ as an orthogonal polynomial and apply its recurrence relation. Specifically:

$$q_{n,r,\sigma}(w) = K^* P_{n-r}^{(0,2r+\sigma+1)}(1-2w),$$

where $P_{n-r}^{(0,2r+\sigma+1)}(1-2w)$ is an orthogonal polynomial with the following recurrence relation:

$$P_{n-r}^{(0,2r+\sigma+1)}(1-2w) = [A_{n-r}(1-2w) + B_{n-r}] \times P_{n-r-1}^{(0,2r+\sigma+1)}(1-2w) - C_{n-r} P_{n-r-2}^{(0,2r+\sigma+1)}(1-2w). \tag{22}$$

Substitute equation (22) into the basic equation (8), replacing $q_{n,r,\sigma}(w)$ with $K^*P_{n-r}^{(0,2r+\sigma+1)}(1-2w)$, to get:

$$\begin{aligned} P_{n,r}^{(\alpha,\beta,\gamma)}(u,v,w) &= P_r^{(\alpha,\beta)}\left(\frac{u}{1-w}\right)(1-w)^r K^* \\ &\times [A_{n-r}(1-2w)+B_{n-r}]\,P_{n-r-1}^{(0,2r+\sigma+1)}(1-2w) \\ &- C_{n-r}P_r^{(\alpha,\beta)}\left(\frac{u}{1-w}\right)(1-w)^r K^* \\ &\times P_{n-r-2}^{(0,2r+\sigma+1)}(1-2w). \end{aligned} \tag{23}$$

After simplifying, we have:

$$\begin{aligned} P_{n,r}^{(\alpha,\beta,\gamma)}(u,v,w) &= P_r^{(\alpha,\beta)}\left(\frac{u}{1-w}\right)(1-w)^r \\ &\times \left\{K^*P_{n-r-1}^{(0,2r+\sigma+1)}(1-2w)\right\}(A_{n-r}(1-2w)+B_{n-r}) \\ &- C_{n-r}P_r^{(\alpha,\beta)}\left(\frac{u}{1-w}\right)(1-w)^r \\ &\times \left\{K^*P_{n-r-2}^{(0,2r+\sigma+1)}(1-2w)\right\}. \end{aligned} \tag{24}$$

Now replace the following:

$$K^*P_{n-r-1}^{(0,2r+\sigma+1)}(1-2w) = q_{n-1,r,\sigma}(w),$$
$$K^*P_{n-r-2}^{(0,2r+\sigma+1)}(1-2w) = q_{n-2,r,\sigma}(w),$$

to obtain:

$$\begin{aligned} P_{n,r}^{(\alpha,\beta,\gamma)}(u,v,w) &= P_r^{(\alpha,\beta)}\left(\frac{u}{1-w}\right)(1-w)^r \\ &\times q_{n-1,r,\sigma}(w)\,(A_{n-r}(1-2w)+B_{n-r}) \\ &- C_{n-r}P_r^{(\alpha,\beta)}\left(\frac{u}{1-w}\right)(1-w)^r q_{n-2,r,\sigma}(w). \end{aligned} \tag{25}$$

Since we know

$$P_{n-1,r}^{(\alpha,\beta,\gamma)}(u,v,w) = P_r^{(\alpha,\beta)}\left(\frac{u}{1-w}\right)(1-w)^r q_{n-1,r,\sigma}(w)$$
$$P_{n-2,r}^{(\alpha,\beta,\gamma)}(u,v,w) = P_r^{(\alpha,\beta)}\left(\frac{u}{1-w}\right)(1-w)^r q_{n-2,r,\sigma}(w)$$

substitute ttheminto equation (25) to obtain the recurrence relation for $P_{n,r}^{(\alpha,\beta,\gamma)}(u,v,w)$ when $\alpha,\beta,\gamma > -1$ and $r = 0,1,\dots,n-2$:

$$\begin{aligned} P_{n,r}^{(\alpha,\beta,\gamma)}(u,v,w) &= (A_{n-r}(1-2w)+B_{n-r}) \\ &\times P_{n-1,r}^{(\alpha,\beta,\gamma)}(u,v,w) - C_{n-r}P_{n-2,r}^{(\alpha,\beta,\gamma)}(u,v,w), \end{aligned} \tag{26}$$

where A_{n-r}, B_{n-r}, and C_{n-r} are constants defined in the previous section. Thus, the recurrence relations for all three cases are now complete.

RECURRENCE RELATION WHEN $w = 0$

In this section, we discuss the case where the recurrence relation becomes univariate when $w = 0$.

The general form of the polynomial is given by:

$$P_{n,r}^{(\alpha,\beta,\gamma)}(u,v,w) = P_r^{(\alpha,\beta)}\left(\frac{u}{1-w}\right)(1-w)^r q_{n,r,\sigma}(w), \tag{27}$$

$r = 0,1,2,\dots,n$, where

$$q_{n,r,\sigma}(w) = \sum_{j=0}^{n-r}(-1)^j\binom{n+r+\sigma+1}{j}b_j^{n-r}(w,u+v). \tag{28}$$

Since our work is defined on the triangular domain:

$$T = \{(u,v,w) : u,v,w \geq 0,\, u+v+w = 1\},$$

we have $u+v = 1-w$, which allows us to express $b_j^{n-r}(w,u+v)$ as:

$$b_j^{n-r}(w,u+v) = \binom{n-r}{j}w^j(1-w)^{n-r-j}. \tag{29}$$

Substitute $q_{n,r,\sigma}(w)$ from equation (29) into equation (27) to obtain:

$$\begin{aligned} P_{n,r}^{(\alpha,\beta,\gamma)}(u,v,w) &= P_r^{(\alpha,\beta)}\left(\frac{u}{1-w}\right)(1-w)^r \\ &\times \sum_{j=0}^{n-r}(-1)^j\binom{n+r+\sigma+1}{j} \\ &\times \binom{n-r}{j}w^j(1-w)^{n-r-j}. \end{aligned} \tag{30}$$

Separate the summation in equation (30) into two parts: when $j = 0$, and when $j = 1,2,\dots,n-r$, as follows:

$$\begin{aligned} P_{n,r}^{(\alpha,\beta,\gamma)}(u,v,w) &= P_r^{(\alpha,\beta)}\left(\frac{u}{1-w}\right)(1-w)^r \\ &\times \left\{(-1)^0\binom{n+r+\sigma+1}{0}\binom{n-r}{0}w^0(1-w)^{n-r-0}\right. \\ &\left. + \sum_{j=1}^{n-r}(-1)^j\binom{n+r+\sigma+1}{j}\binom{n-r}{j}w^j(1-w)^{n-r-j}\right\} \end{aligned} \tag{31}$$

The first term in the summation, when $j = 0$, simplifies to:

$$\begin{aligned} (-1)^0\binom{n+r+\sigma+1}{0}\binom{n-r}{0}w^0(1-w)^{n-r} \\ = (1-w)^{n-r}. \end{aligned} \tag{32}$$

Substitute this into the equation to obtain

$$P_{n,r}^{(\alpha,\beta,\gamma)}(u,v,w) = P_r^{(\alpha,\beta)}\left(\frac{u}{1-w}\right)(1-w)^r \times \left\{(1-w)^{n-r} + \sum_{j=1}^{n-r}(-1)^j \binom{n+r+\sigma+1}{j} \times \binom{n-r}{j} w^j (1-w)^{n-r-j}\right\}. \quad (33)$$

- **Case when $w=0$:** Now, set $w=0$ to get:

$$P_{n,r}^{(\alpha,\beta,\gamma)}(u,v,0) = P_r^{(\alpha,\beta)}\left(\frac{u}{1-0}\right)(1-0)^r \times \left\{1 + \sum_{j=1}^{n-r}(-1)^j \binom{n+r+\sigma+1}{j}\binom{n-r}{j}\right\}. \quad (34)$$

Since all terms in the summation for $j=1,2,\ldots,n-r$ vanish (because $0^j=0$ for all $j \geq 1$), the expression reduces to:

$$P_{n,r}^{(\alpha,\beta,\gamma)}(u,v,0) = P_r^{(\alpha,\beta)}(u)(1)^r(1)^{n-r} = P_r^{(\alpha,\beta)}(u). \quad (35)$$

So, the last equation becomes:

$$P_{n,r}^{(\alpha,\beta,\gamma)}(u,v,0) = P_r^{(\alpha,\beta)}\left(\frac{u}{1-0}\right)(1-0)^r\{1+0\}. \quad (36)$$

Simplifying this, we get:

$$P_{n,r}^{(\alpha,\beta,\gamma)}(u,v,0) = P_r^{(\alpha,\beta)}(u). \quad (37)$$

This means that when $w=0$, the polynomial $P_{n,r}^{(\alpha,\beta,\gamma)}(u,v,0)$ becomes a univariate polynomial that depends only on r.

- **Tabular Representation:** Consider the following table:

$$\begin{array}{l} P_{0,0}^{(\alpha,\beta,\gamma)}(u,v,0) \\ P_{1,0}^{(\alpha,\beta,\gamma)}(u,v,0), P_{1,1}^{(\alpha,\beta,\gamma)}(u,v,0) \\ P_{2,0}^{(\alpha,\beta,\gamma)}(u,v,0), P_{2,1}^{(\alpha,\beta,\gamma)}(u,v,0), P_{2,2}^{(\alpha,\beta,\gamma)}(u,v,0) \\ \vdots \\ P_{n,0}^{(\alpha,\beta,\gamma)}(u,v,0), P_{n,1}^{(\alpha,\beta,\gamma)}(u,v,0), \ldots, P_{n,n}^{(\alpha,\beta,\gamma)}(u,v,0) \end{array}$$

- **Consistency Across Rows and Columns:** The first column has the same value for all $r=0$. By equation (35):

$$P_{0,0}^{(\alpha,\beta,\gamma)}(u,v,0) = P_{1,0}^{(\alpha,\beta,\gamma)}(u,v,0) = P_{2,0}^{(\alpha,\beta,\gamma)}(u,v,0) = \ldots = P_{n,0}^{(\alpha,\beta,\gamma)}(u,v,0) = P_0^{(\alpha,\beta)}(u).$$

The polynomials in the second column, for $r=1$, also yield the same values:

$$P_{1,1}^{(\alpha,\beta,\gamma)}(u,v,0) = P_{2,1}^{(\alpha,\beta,\gamma)}(u,v,0) = \ldots = P_{n,1}^{(\alpha,\beta,\gamma)}(u,v,0) = P_1^{(\alpha,\beta)}(u).$$

Following this pattern up to the general case:

$$P_{i,i}^{(\alpha,\beta,\gamma)}(u,v,0) = P_{i+1,i}^{(\alpha,\beta,\gamma)}(u,v,0) = \ldots = P_{n,i}^{(\alpha,\beta,\gamma)}(u,v,0) = P_i^{(\alpha,\beta)}(u),$$

for $n, j > i$.

- **Final Row Interpretation:** This implies that only one row, the last row of the table, remains:

$$P_{n,0}^{(\alpha,\beta,\gamma)}(u,v,0), P_{n,1}^{(\alpha,\beta,\gamma)}(u,v,0), \ldots, P_{n,n}^{(\alpha,\beta,\gamma)}(u,v,0).$$

These polynomials can be replaced respectively by:

$$P_0^{(\alpha,\beta)}(u), P_1^{(\alpha,\beta)}(u), P_2^{(\alpha,\beta)}(u), \ldots, P_n^{(\alpha,\beta)}(u).$$

- **Conclusion:** When $w=0$, all bivariate Jacobi polynomials reduce to a univariate system of Jacobi polynomials of degree r, which can be expressed in the Bernstein representation.

NUMERICAL SIMULATIONS

In this section, cubic Jacobi-weighted orthogonal polynomials are constructed using equation (7). For simplicity, we use the notation $\sigma = \alpha+\beta+\gamma$. To visualize these polynomials, the software *Mathematica* is employed.

We consider the triangular domain:

$$T = \{(u,v,w) : u,v,w \geq 0,\, u+v+w = 1\},$$

with specific values chosen for the parameters α, β, and γ, where w is defined as:

$$w = 1-u-v.$$

The numerical simulations allow us to plot and analyze the behavior of these polynomials under different parameter settings. The results provide insights into the structure and properties of the bivariate Jacobi polynomials, which play a critical role in various applications, including approximation theory and numerical analysis.

FIRST POLYNOMIAL $P_{3,0}^{(\alpha,\beta,\gamma)}(u,v,w)$

The first polynomial in the cubic case, for $r=0$, is given by:

$$P_{3,0}^{(\alpha,\beta,\gamma)}(u,v,w) = (u+v)^3 - 3(4+\sigma)w(u+v)^2 + \frac{3}{2}(4+\sigma)(3+\sigma)w^2(u+v) - \frac{1}{6}(4+\sigma)(3+\sigma)(2+\sigma)w^3, \quad (38)$$

where $\sigma = \alpha+\beta+\gamma$.

Note. All figures in this section have been revised to include axis labels, colormap descriptions, and quantitative captions for improved scientific readability.

In the following cases, different values of α, β, and γ are considered:

- **Case 1:** $\alpha=\beta=\gamma=0$

$$P_{3,0}^{(0,0,0)}(u,v,w) = (u+v)^3 - 12w(u+v)^2 + 18w^2(u+v) - 4w^3.$$

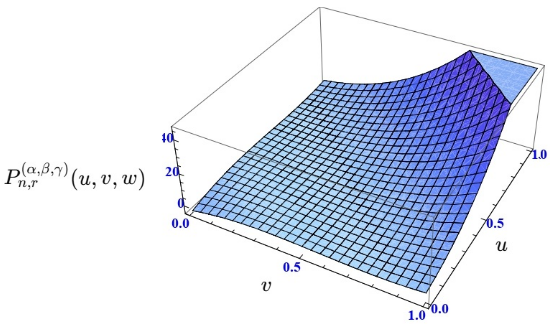

Fig. 1. *Surface plot of $P_{3,0}^{(0,0,0)}(u,v,w)$ over the triangular domain $u+v+w=1$, with $\alpha=\beta=\gamma=0$. Axes represent u, v, and polynomial value. The vertical axis shows values approximately from 0 to 40. A standard colormap illustrates intensity.*

- **Case 2:** $\alpha=\beta=\gamma=-0.5$

$$P_{3,0}^{(-0.5,-0.5,-0.5)}(u,v,w) = (u+v)^3 - 7.5w(u+v)^2 + 5.625w^2(u+v) - 0.3125w^3.$$

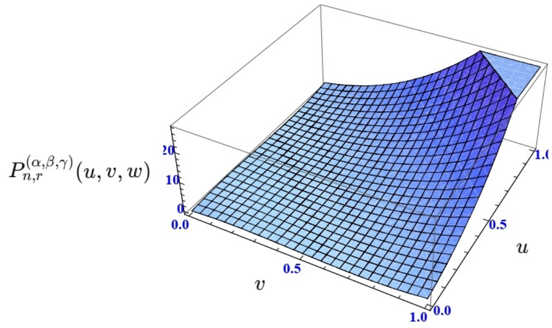

Fig. 2. *Surface plot of $P_{3,0}^{(-0.5,-0.5,-0.5)}(u,v,w)$ for $\alpha=\beta=\gamma=-0.5$. The plot shows values between 0 and 20 with labeled axes u, v, and polynomial height.*

- **Case 3:** $\alpha=\beta=\gamma=0.5$

$$P_{3,0}^{(0.5,0.5,0.5)}(u,v,w) = (u+v)^3 - 16.5w(u+v)^2 + 37.125w^2(u+v) - 14.4375w^3.$$

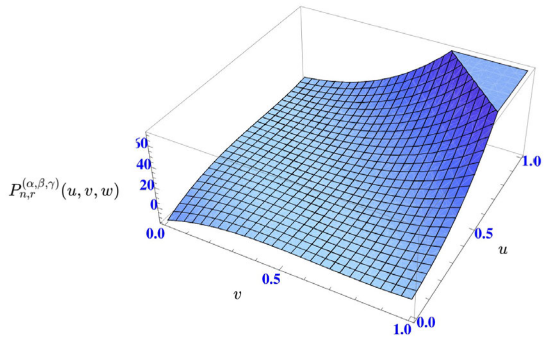

Fig. 3. *Visualization of $P_{3,0}^{(0.5,0.5,0.5)}(u,v,w)$ with $\alpha=\beta=\gamma=0.5$. Polynomial values range up to 60. Axes u, v, and height are clearly defined.*

- **Case 4:** $\alpha=0.5$, $\beta=\gamma=-0.5$

$$P_{3,0}^{(0.5,-0.5,-0.5)}(u,v,w) = (u+v)^3 - 10.5w(u+v)^2 + 13.125w^2(u+v) - 2.1875w^3.$$

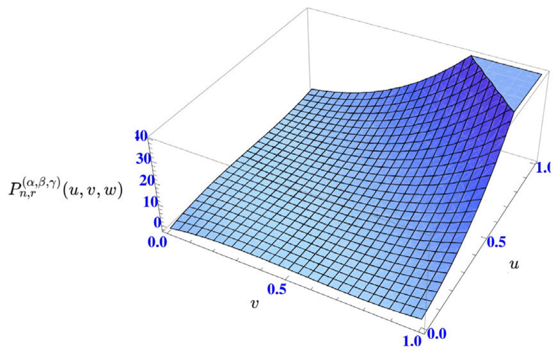

Fig. 4. *Plot of $P_{3,0}^{(0.5,-0.5,-0.5)}(u,v,w)$ for $\alpha=0.5$, $\beta=\gamma=-0.5$. Polynomial surface is shown with intensity via colormap. Axes denote u, v, and output value.*

- **Case 5:** $\alpha = \frac{1}{3}$, $\beta = -\frac{2}{5}$, $\gamma = 0$

$$P_{3,0}^{\left(\frac{1}{3},-\frac{2}{5},0\right)}(u,v,w) = (u+v)^3 - 11.8w(u+v)^2 + 17.3067w^2(u+v) - 3.7177w^3.$$

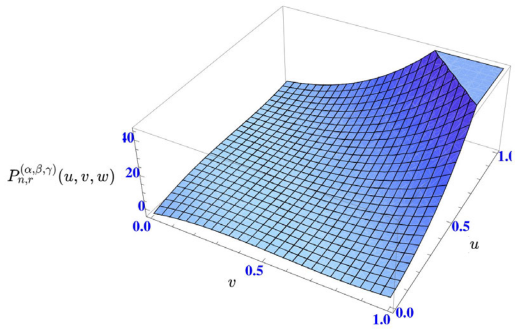

Fig. 5. *Surface plot of* $P_{3,0}^{\left(\frac{1}{3},-\frac{2}{5},0\right)}(u,v,w)$ *for* $\alpha = \frac{1}{3}$, $\beta = -\frac{2}{5}$, $\gamma = 0$. *Polynomial values shown between 0 and 40.*

SECOND POLYNOMIAL $P_{3,1}^{(\alpha,\beta,\gamma)}(u,v,w)$

The second polynomial in the cubic case, when $r = 1$, is given by:

$$P_{3,1}^{(\alpha,\beta,\gamma)}(u,v,w) = [(1+\alpha)u - (1+\beta)v]\Big[(u+v)^2 - 2(5+\sigma)w(u+v) + \frac{1}{2}(5+\sigma)(4+\sigma)w^2\Big]$$

where $\sigma = \alpha + \beta + \gamma$.

This polynomial is visualized using *Mathematica* in several cases for different values of α, β, and γ as follows:

- **Case 1:** $\alpha = \beta = \gamma = 0$

$$P_{3,1}^{(0,0,0)}(u,v,w) = (u-v) \times \left[(u+v)^2 - 10w(u+v) + 10w^2\right].$$

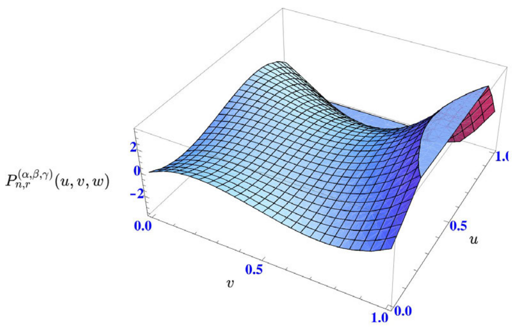

Fig. 6. *Plot of* $P_{3,1}^{(0,0,0)}(u,v,w)$ *with* $\alpha = \beta = \gamma = 0$. *Symmetric structure about* $u = v$; *axes and colormap included.*

- **Case 2:** $\alpha = \beta = \gamma = -0.5$

$$P_{3,1}^{(-0.5,-0.5,-0.5)}(u,v,w) = 0.5(u-v) \times \left[(u+v)^2 - 7w(u+v) + 4.375w^2\right].$$

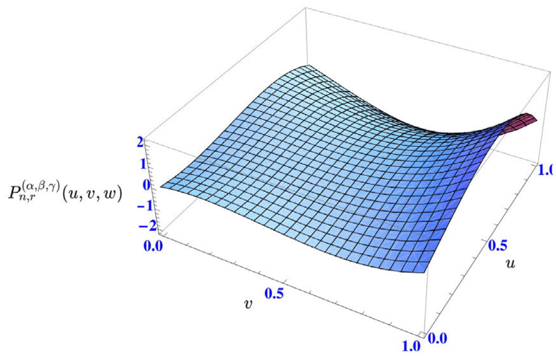

Fig. 7. *Visualization of* $P_{3,1}^{(-0.5,-0.5,-0.5)}(u,v,w)$ *for* $\alpha = \beta = \gamma = -0.5$. *Values oscillate around zero with small amplitude. Axes and color scale are labeled.*

- **Case 3:** $\alpha = \beta = \gamma = 0.5$

$$P_{3,1}^{(0.5,0.5,0.5)}(u,v,w) = 1.5(u-v) \times \left[(u+v)^2 - 13w(u+v) + 17.875w^2\right].$$

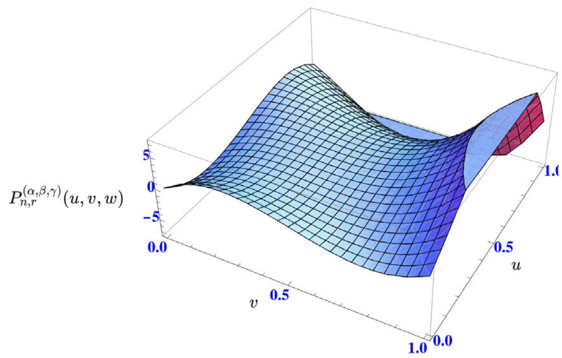

Fig. 8. *Surface plot of* $P_{3,1}^{(0.5,0.5,0.5)}(u,v,w)$ *for* $\alpha = \beta = \gamma = 0.5$. *The polynomial shows a steep slope and stronger asymmetry. Axes are labeled u, v, and polynomial height; colormap represents magnitude.*

- **Case 4:** $\alpha = 0.5$, $\beta = \gamma = -0.5$

$$P_{3,1}^{(0.5,-0.5,-0.5)}(u,v,w) = (1.5u - 0.5v) \times \left[(u+v)^2 - 9w(u+v) + 7.875w^2\right].$$

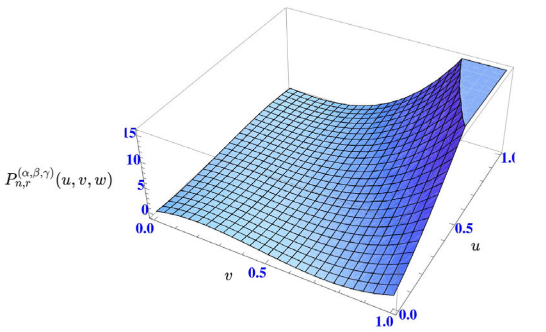

Fig. 9. *Surface plot of* $P_{3,1}^{(0.5,-0.5,-0.5)}(u,v,w)$ *with* $\alpha = 0.5$, $\beta = \gamma = -0.5$. *The polynomial exhibits a tilted saddle structure with labeled axes and color gradient.*

– **Case 5:** $\alpha = \frac{1}{3}$, $\beta = -\frac{2}{5}$, $\gamma = 0$

$$P_{3,1}^{(\frac{1}{3},-\frac{2}{5},0)}(u,v,w) = (1.34u - 0.6v) \times \left[(u+v)^2 - 9.867w(u+v) + 9.834w^2\right].$$

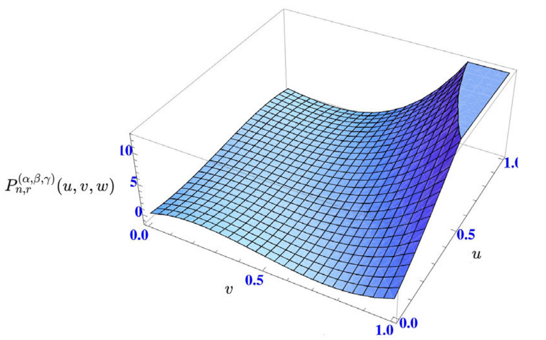

Fig. 10. *Surface plot of* $P_{3,1}^{(\frac{1}{3},-\frac{2}{5},0)}(u,v,w)$ *for* $\alpha = \frac{1}{3}$, $\beta = -\frac{2}{5}$, $\gamma = 0$. *The structure is asymmetric with polynomial values ranging up to 10. All axes are labeled.*

THIRD POLYNOMIAL $P_{3,2}^{(\alpha,\beta,\gamma)}(u,v,w)$

The third polynomial in the cubic case, for $r = 2$, is given by:

$$P_{3,2}^{(\alpha,\beta,\gamma)}(u,v,w) = \left[\frac{1}{2}(2+\beta)(1+\beta)v^2 - (2+\alpha)(2+\beta)uv + \frac{1}{2}(2+\alpha)(1+\alpha)u^2\right] \times [(u+v) - (6+\sigma)w],$$

where $\sigma = \alpha + \beta + \gamma$.

The same cases are considered to plot this polynomial, with different values of α, β, and γ:

– **Case 1:** $\alpha = \beta = \gamma = 0$

$$P_{3,2}^{(0,0,0)}(u,v,w) = [v^2 - 4uv + u^2][(u+v) - 6w].$$

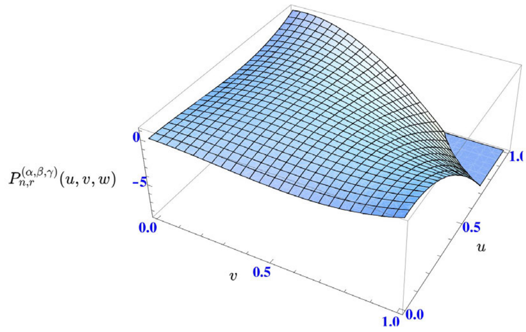

Fig. 11. *Surface plot of* $P_{3,2}^{(0,0,0)}(u,v,w)$ *for* $\alpha = \beta = \gamma = 0$. *The plot shows a quadratic difference pattern multiplied by a linear w-term. Axes u, v, and height are labeled.*

– **Case 2:** $\alpha = \beta = \gamma = -0.5$

$$P_{3,2}^{(-0.5,-0.5,-0.5)}(u,v,w) = [0.375v^2 - 2.25uv + 0.375u^2][(u+v) - 4.5w].$$

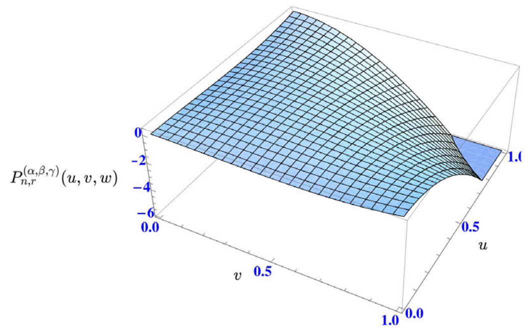

Fig. 12. *Surface plot of* $P_{3,2}^{(-0.5,-0.5,-0.5)}(u,v,w)$ *with* $\alpha = \beta = \gamma = -0.5$. *The polynomial displays gentle curvature with a linear w-modulation. Axes and colormap are defined.*

– **Case 3:** $\alpha = \beta = \gamma = 0.5$

$$P_{3,2}^{(0.5,0.5,0.5)}(u,v,w) = [1.875v^2 - 6.25uv + 1.875u^2][(u+v) - 7.5w].$$

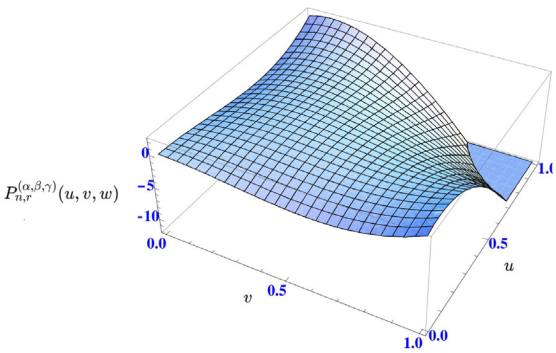

Fig. 13. *Surface plot of* $P_{3,2}^{(0.5,0.5,0.5)}(u,v,w)$ *for* $\alpha = \beta = \gamma = 0.5$. *The curvature is sharper with stronger decay toward* $w = 1$. *All axes are labeled and a smooth color map is used.*

– **Case 4:** $\alpha = 0.5$, $\beta = \gamma = -0.5$

$$P_{3,2}^{(0.5,-0.5,-0.5)}(u,v,w) = [0.375v^2 - 3.75uv + 1.875u^2][(u+v) - 5.5w].$$

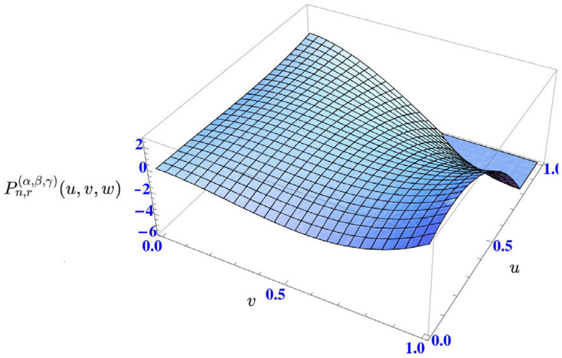

Fig. 14. *Surface plot of* $P_{3,2}^{(0.5,-0.5,-0.5)}(u,v,w)$ *with* $\alpha = 0.5$*,* $\beta = \gamma = -0.5$*. The plot shows asymmetry and varying concavity across the triangle. Axes u, v, and function value are labeled.*

– **Case 5:** $\alpha = \frac{1}{3}$, $\beta = -\frac{2}{5}$, $\gamma = 0$

$$P_{3,2}^{(\frac{1}{3},-\frac{2}{5},0)}(u,v,w) = [0.96v^2 - 3.734uv + 1.556u^2] \times [(u+v) - 5.934w].$$

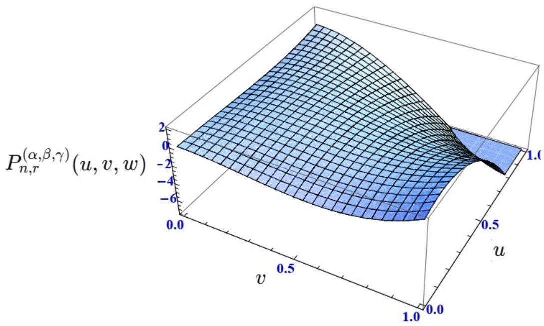

Fig. 15. *Surface plot of* $P_{3,2}^{(\frac{1}{3},-\frac{2}{5},0)}(u,v,w)$ *for* $\alpha = \frac{1}{3}$*,* $\beta = -\frac{2}{5}$*,* $\gamma = 0$*. The polynomial demonstrates mixed curvature with axis-aligned variation. Axes and colorbar are defined.*

FOURTH POLYNOMIAL $P_{3,3}^{(\alpha,\beta,\gamma)}(u,v,w)$

The last polynomial in the cubic case, when $r = 3$, is given by:

$$P_{3,3}^{(\alpha,\beta,\gamma)}(u,v,w) = -\frac{1}{6}(3+\beta)(2+\beta)(1+\beta)v^3 + \frac{1}{2}(3+\alpha)(3+\beta)(2+\beta)uv^2$$
$$\frac{1}{2}(3+\alpha)(2+\alpha)(3+\beta)u^2v + \frac{1}{6}(3+\alpha)(2+\alpha)(1+\alpha)u^3,$$

where $\sigma = \alpha + \beta + \gamma$.

As before, five cases are considered for different values of α, β, and γ to generate these polynomials:

– **Case 1:** $\alpha = \beta = \gamma = 0$

$$P_{3,3}^{(0,0,0)}(u,v,w) = -v^3 + 9uv^2 - 9u^2v + u^3.$$

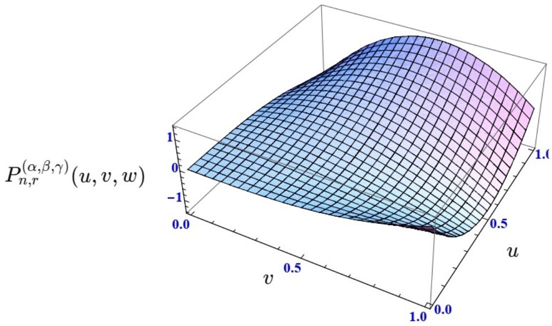

Fig. 16. *Surface plot of* $P_{3,3}^{(0,0,0)}(u,v,w)$ *for* $\alpha = \beta = \gamma = 0$*. This is a skew–symmetric cubic in u and v without dependence on w. All axes are labled and values vary from -1 to 1.*

– **Case 2:** $\alpha = \beta = \gamma = -0.5$

$$P_{3,3}^{(-0.5,-0.5,-0.5)}(u,v,w) = -0.3125v^3 + 4.6875uv^2 - 4.6875u^2v + 0.3125u^3.$$

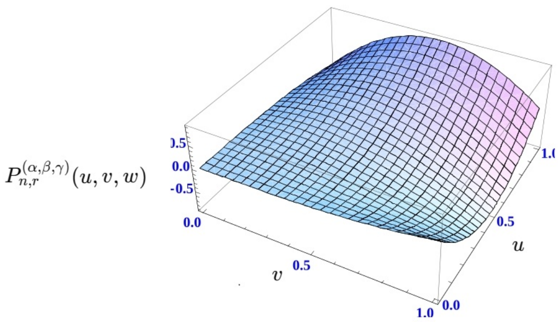

Fig. 17. *Surface plot of* $P_{3,3}^{(-0.5,-0.5,-0.5)}(u,v,w)$ *for* $\alpha = \beta = \gamma = -0.5$*. A softened version of the skew-cubic structure appears. Axes u, v, and height are labeled.*

– **Case 3:** $\alpha = \beta = \gamma = 0.5$

$$P_{3,3}^{(0.5,0.5,0.5)}(u,v,w) = -2.1875v^3 + 15.3125uv^2 - 15.3125u^2v + 2.1875u^3.$$

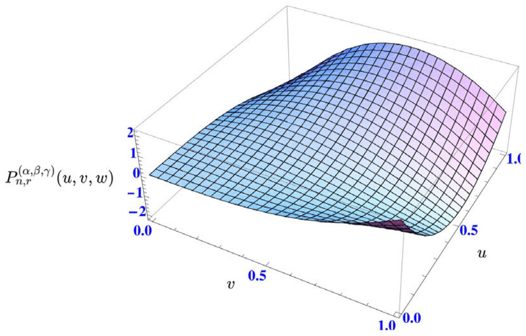

Fig. 18. *Surface plot of* $P_{3,3}^{(0.5,0.5,0.5)}(u,v,w)$ *for* $\alpha = \beta = \gamma = 0.5$. *This polynomial shows a pronounced peak-valley configuration in the triangular domain. Axes and color scale are included.*

– **Case 4:** $\alpha = 0.5$**,** $\beta = \gamma = -0.5$

$$P_{3,3}^{(0.5,-0.5,-0.5)}(u,v,w) = -0.3125v^3 + 6.5625uv^2 - 10.9375u^2v + 2.1875u^3.$$

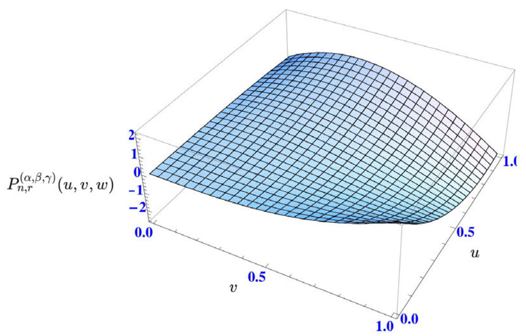

Fig. 19. *Surface plot of* $P_{3,3}^{(0.5,-0.5,-0.5)}(u,v,w)$ *with* $\alpha = 0.5$, $\beta = \gamma = -0.5$. *The function is steeper in u-direction and asymmetric. Axes u, v, and z are labeled.*

– **Case 5:** $\alpha = \frac{1}{3}$**,** $\beta = -\frac{2}{5}$**,** $\gamma = 0$

$$P_{3,3}^{(\frac{1}{3},-\frac{2}{5},0)}(u,v,w) = -0.416v^3 + 6.934uv^2 - 10.11u^2v + 1.038u^3.$$

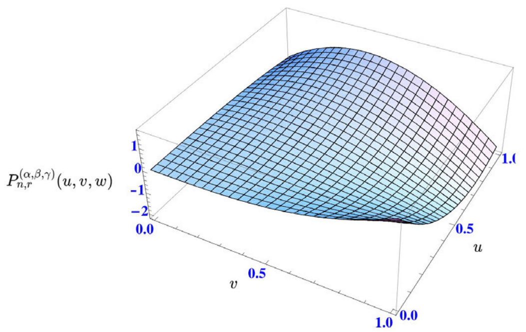

Fig. 20. *Surface plot of* $P_{3,3}^{(\frac{1}{3},-\frac{2}{5},0)}(u,v,w)$ *for* $\alpha = \frac{1}{3}$, $\beta = -\frac{2}{5}$, $\gamma = 0$. *The polynomial shows a mildly skewed cubic profile. Axes and colormap are properly labeled.*

CONCLUSION

In this work, we constructed and analyzed the cubic Jacobi-weighted orthogonal polynomials $P_{3,r}^{(\alpha,\beta,\gamma)}(u,v,w)$ for $r = 0,1,2,3$ over the triangular domain $T = \{(u,v,w) : u,v,w \geq 0, u+v+w = 1\}$. By considering various parameter values for α, β, and γ, we derived explicit polynomial expressions and visualized them using *Mathematica*. The polynomial structures exhibit significant variations depending on the chosen parameter values, influencing both the magnitude and shape of the resulting polynomials. The visualizations provide valuable insights into the behavior of these polynomials across different scenarios, demonstrating their utility in applications that involve polynomial approximations on triangular domains. The constructed recurrence relations also offer an efficient way to compute these polynomials without the need for direct summation formulas, making them more practical for computational tasks. However, in this work, we do not claim the uniqueness or optimality of the derived recurrence relations when compared to other polynomial bases on simplices such as Dubiner or orthonormal Bernstein bases. These comparisons are left as an important direction for future research. In the same regard, it is worth mention that although the derived recurrence relations are expected to offer computational advantages over classical summation formulas, this paper does not include empirical benchmarks. Evaluating runtime efficiency on large-scale inputs (e.g., 10^5 tuples) and comparing recurrence-based vs. classical construction remains an important direction for future research. In addition, a full computational implementation of the proposed recurrence relations using frameworks such as NumPy or TensorFlow, along with a scalability analysis, will be addressed in a future extension of this work. A practical appendix containing code and numerical benchmarks is also planned.

REFERENCES

Alshanti WG, Batiha IM, Alshanty A, Zraiqat A, Jebril IH, Abu Ahmad M (2023). Perturbed trapezoid like inequalities. Sci Technol Indones 8(2):205–11.

Anakira NR, Almalki A, Katatbeh D, Hani GB, Jameel AF, Al Kalbani KS, Abu-Dawas M (2023). An algorithm for solving linear and non-linear Volterra integro-differential equations. Int J Adv Soft Comput Appl 15(3):77–83.

Batiha IM (2011). Restriction method for approximating square roots. Int J Open Probl Comput Math 4(3):146–51.

Batiha IM, Jebril I, Alshorm S, Al-Nana AA (2023). Some results on zeros of the monic polynomial of the Frobenius companion matrix. Advances in Fixed Point Theory 13(1):13.

Batiha IM, Jebril IH, Alshorm S, Anakira N, Alkhazaleh S (2024). On generalized matrix Mittag-Leffler function. IAENG International Journal of Applied Mathematics 54(3):576–80.

Berir M (2024). Analysis of the effect of white noise on the Halvorsen system of variable-order fractional derivatives using a novel numerical method. Int J Adv Soft Comput Appl 16(3):294–306.

Dunkl CF, Xu Y (2014). *Orthogonal Polynomials of Several Variables* (2nd ed.). Cambridge: Cambridge University Press.

Farraj G, Maayah B, Khalil R, Beghami W (2023). An algorithm for solving fractional differential equations using conformable optimized decomposition method. Int J Adv Soft Comput Appl 15(1):187–96.

Hajaj RI, Batiha IM, Aljazzazi M, Jebril IH, Bouchenak A, Fakhreddine S, Batiha B (2025). On stability analysis of nonlinear systems. IAENG International Journal of Applied Mathematics 55(4):873–78.

Hawawsheh L, Qazza A, Saadeh R, Zraqat A, Batiha IM (2023). L^p-mapping properties of a class of spherical integral operators. Axioms 12(9):802.

Koekoek R, Lesky PA, Swarttouw RF (2010). *Hypergeometric Orthogonal Polynomials and Their q-Analogues*. Berlin: Springer, 578 pp.

Koornwinder TH (1975). Two-variable analogues of the classical orthogonal polynomials. In Askey RA (Ed.), *Theory and Application of Special Functions* (pp. 435–495). New York: Academic Press.

Merabti NL, Batiha IM, Rezzoug I, Ouannas A, Ouassaeif TE (2023). On sentinel method of one-phase Stefan problem. J Nig Soc Phys Sci 5:1772.

Qawaqneh H (2024). Functional analysis with fractional operators: properties and applications in metric spaces. Adv Fixed Point Theory 14:Article ID 62.

Rababah A (2003). Transformation of Chebyshev-Bernstein polynomial basis. Computational Methods in Applied Mathematics 3(4):608–22.

Rababah A (2004). Jacobi-Bernstein basis transformation. Computational Methods in Applied Mathematics 4(2):206–14.

Rababah A (2005). Bivariate orthogonal polynomials on triangular domains. Mathematics and Computers in Simulation 78:107–11.

Rababah A, Alqudah M (2005). Jacobi-weighted orthogonal polynomials on triangular domains. Journal of Applied Mathematics 3:205–17.

Szegő G (1975). *Orthogonal Polynomials* (4th ed.). Providence, RI: American Mathematical Society.