

PRECISION OF THE INVARIATOR ESTIMATOR OF THE SURFACE AREA OF AN ELLIPSOID

LUIS M. CRUZ-ORIVE

University of Cantabria, (Ret.), E-39005 Santander, Spain
e-mail: luis.cruz@unican.es

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ABSTRACT

Consider a triaxial ellipsoid K of surface area S . Fix an arbitrary point P in the interior of K , that is $P \in K^\circ$, and generate a sectioning plane $L_{2[P]}^3$ through P , whose normal direction u is uniform random on the unit hemisphere $\mathbb{S}_+^2$. In the ellipse of section $K \cap L_{2[P]}^3$, let M and m denote the lengths of the major and minor principal semiaxes, respectively, and let r denote the distance of P from the ellipse centre. Then $\hat{S} = 2\pi(M^2 + m^2 + r^2)$ is an unbiased estimator of S . The purpose of this paper is to express $\hat{S}$ in terms of the eight parameters involved, namely the lengths of the three principal semiaxes of K , the three Cartesian coordinates of P , and the two spherical polar coordinates (ϕ, θ) of u . Then $\text{Var}(\hat{S})$ is accessible via a double definite integral in (ϕ, θ) which can be evaluated quickly with available software for any K and any choice of $P \in K^\circ$.

Keywords: Elliptic integrals, ellipsoidal section, flower formula, heat map, integral geometry, invariator, pivotal plane, pivotal point, stereology, support set.

INTRODUCTION

The main purpose of this paper is to contribute to the theory of the estimator $\hat{S}$, described in the Abstract, of the surface area S of an ellipsoid. The estimator $\hat{S}$ was obtained by Cruz-Orive (2011), and it is based on the invariator principle of stereology, (Cruz-Orive, 2005; Gual-Arnau and Cruz-Orive, 2009; Auneau and Jensen, 2010; Gual-Arnau et al., 2010; Thórisdóttir and Kiderlen, 2014; Cruz-Orive and Gual-Arnau, 2015; Jensen and Kiderlen, 2017; Cruz-Orive, 2024, p. 49). The background of the invariator is summarized in the next section. Note that $\hat{S}$ depends solely on three measurements made in a planar section. A formula is obtained for $\hat{S}$ in terms of the eight parameters mentioned in the Abstract - the corresponding expression of $\text{Var}(\hat{S})$ depends on a definite double integral whose numerical value can be evaluated efficiently with available software knowing the lengths of the principal semiaxes of the ellipsoid, and the coordinates of a pivotal point P in the ellipsoid. To illustrate this, the coefficient of variation $\text{CV}(\hat{S}) = \sqrt{\text{Var}(\hat{S})}/S$ is computed for an ellipsoid of semiaxes lengths $\{1, 2, 3\}$ as the pivotal point P varies in a fine point grid within each of the three principal ellipses of the ellipsoid. The results are represented by means of heat maps to give a visual impression of the variation of the accuracy of $\hat{S}$ as P varies in different regions of the ellipsoid.

The question of whether the estimator $\hat{S}$ can be generalized to $\mathbb{R}^d$ for $d > 3$ is briefly discussed in the last section. The potential application of $\hat{S}$ is also discussed there. The latter is not the main focus of this paper, however, because in nature no particles exist which are perfect ellipsoids.

A BRIEF REVIEW OF THE INVARIATOR ESTIMATION OF THE SURFACE AREA OF A CONVEX BODY

Consider a convex body $K \subset \mathbb{R}^3$ of surface area S , plus a point $P \in K^\circ$, called a pivotal point, and an isotropically oriented plane $L_{2[P]}^3$ through P , called a pivotal plane. 'Isotropically oriented' means that the direction $u \in \mathbb{S}_+^2$ of an axis normal to $L_{2[P]}^3$ is uniform random on the unit hemisphere $\mathbb{S}_+^2$. Any transect $C_u := K \cap L_{2[P]}^3$ is convex, (hereafter, " $A := B$ " means " A is defined by B ", or " B is denoted by A "). Let H_{C_u} denote the support set or 'flower' of C_u , namely the domain enclosed by the graph of the support function of C_u with origin P . The support function is,

$$h(u, \omega) = \sup(x \cdot e_\omega : x \in C_u), \omega \in [0, 2\pi), \quad (1)$$

where e_ω is a unit vector of direction ω in the pivotal plane, and x is the radius vector of a point in C_u with origin P - Fig. 3 illustrates the case in which C_u is an

ellipse. Let $A(\cdot)$ denote the area of a planar set. The 'flower formula' for S reads,

$$S = \frac{2}{\pi} \int_{\mathbb{S}_+^2} A(H_{C_u}) du = 4\mathbb{E}(A(H_{C_u})), \quad (2)$$

the expectation being with respect to (w.r.t.) the uniform probability element $\mathbb{P}(du) = du/(2\pi)$. Therefore,

$$\widehat{S}(u) = 4A(H_{C_u}) \quad (3)$$

is an unbiased estimator (UE) of S , (Cruz-Orive, 2005). The estimator $\widehat{S}(u)$, and its variance, depend on the position of $P \in K^\circ$, but its expectation does not. (To simplify the notation, the dependence of $\widehat{S}(u)$, C_u , etc., on P is ignored in the sequel). The flower area is,

$$A(H_{C_u}) = \pi \mathbb{E}(h^2(u, \omega)), \quad (4)$$

the expectation being w.r.t. $\mathbb{P}(d\omega) = d\omega/(2\pi)$. Thus,

$$\widehat{S}(u, \omega) = 4\pi h^2(u, \omega) \quad (5)$$

is another UE of the surface area of the convex body K . Based on the critical points of ∂C_u , the preceding estimator was generalized for non convex particles by Thórisdóttir and Kiderlen (2014) and Thórisdóttir et al. (2014) using Morse theory. Gual-Arnau and CruzOrive (2016) obtained the same estimator by a different approach, and its simplified version was called the 'peak-and-valley formula' in Cruz-Orive and Gual-Arnau (2015), see also Cruz-Orive (2024), p. 50.

If K is an ellipsoid, then the transect C_u is an ellipse of principal semiaxes $0 < m(u) \leq M(u)$, say, and $A(H_{C_u})$ can be computed in terms of $m(u)$, $M(u)$, and the distance $r(u)$ of the pivotal point $P \in C_u$ from the ellipse centre, see Fig. 1b,c. The result is,

$$A(H_{C_u}) = \frac{\pi}{2} (M^2(u) + m^2(u) + r^2(u)), \quad (6)$$

(Appendix A.1), and, by Eq. 3,

$$\widehat{S}(u) = 2\pi (M^2(u) + m^2(u) + r^2(u)) \quad (7)$$

is unbiased for S .

For a convex body, $\mathbb{E}(\widehat{S}(u, \omega) | u) = \widehat{S}(u)$, where $\widehat{S}(u)$ is given by Eq. 3, or by Eq. 7 for an ellipsoid. Therefore, in either case,

$$\begin{aligned} \text{Var}(\widehat{S}(u, \omega)) \\ = \text{Var}\{\mathbb{E}(\widehat{S}(u, \omega) | u)\} + \mathbb{E}\{\text{Var}(\widehat{S}(u, \omega) | u)\} \\ \geq \text{Var}(\widehat{S}(u)). \end{aligned} \quad (8)$$

Cruz-Orive (2008) studied the variances of various invariator estimators of surface area and volume,

and compared them with those of classical local estimators using a sphere model. The variance of $\widehat{S}(\omega, u)$, was considered by Dvořák and Jensen (2013), who obtained explicit results for spheroids (namely biaxial ellipsoids). The variance of the aforementioned generalization of Eq. 5 was studied by Thórisdóttir et al. (2014).

THE INVARIATOR ESTIMATOR OF THE SURFACE AREA S OF AN ELLIPSOID

THE SURFACE AREA OF AN ELLIPSOID

Consider an ellipsoid $K \subset \mathbb{R}^3$ of principal semiaxes $0 < a_1 \leq a_2 \leq a_3 < \infty$. In Cartesian coordinates,

$$K := \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 : \sum_{i=1}^3 x_i^2/a_i^2 \leq 1 \right\}. \quad (9)$$

The surface area of K is,

$$\begin{aligned} S &:= S(a_1, a_2, a_3) \\ &= 2\pi a_1^2 + \frac{2\pi a_2 a_3}{\sin(\varphi)} \cdot (E(\varphi, k) \sin^2(\varphi) \\ &\quad + F(\varphi, k) \cos^2(\varphi)), \end{aligned} \quad (10)$$

where

$$\cos(\varphi) = \frac{a_1}{a_3}, \quad k^2 = \frac{1 - (a_1/a_2)^2}{1 - (a_1/a_3)^2}, \quad (11)$$

(Legendre, 1811, p. 191), and

$$\begin{aligned} F(\varphi, k) &= \int_0^\varphi \frac{dt}{\sqrt{1 - k^2 \sin^2(t)}} \\ E(\varphi, k) &= \int_0^\varphi \sqrt{1 - k^2 \sin^2(t)} dt \end{aligned} \quad (12)$$

represent the incomplete elliptic integrals of the first and second kind, respectively.

PARAMETRIZATION OF THE PIVOTAL POINT AND THE PIVOTAL PLANE

We introduce the following definitions and notation.

- Principal semiaxes lengths of the ellipsoid K :

$$a := (a_1, a_2, a_3), (0 < a_1 \leq a_2 \leq a_3 < \infty). \quad (13)$$

- Cartesian coordinates of a pivotal point $P(x_0) \in K^\circ$:

$$x_0 := (x_{01}, x_{02}, x_{03}), \left(\sum_{i=1}^3 x_{0i}^2/a_i^2 - 1 < 0 \right). \quad (14)$$

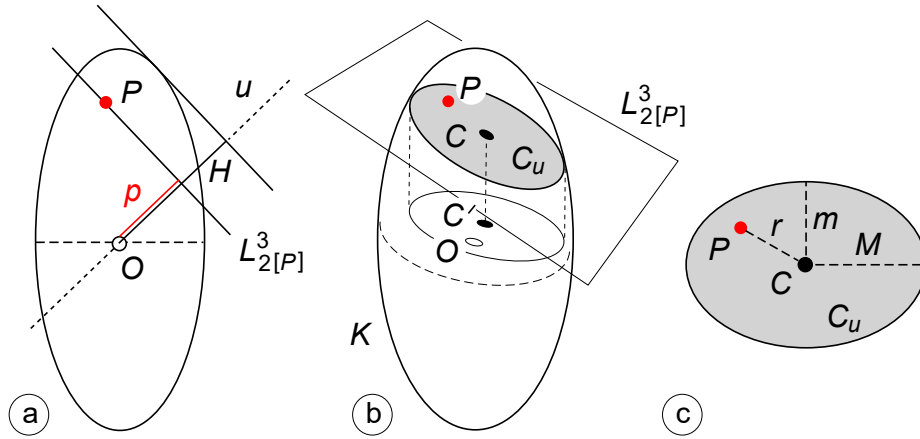


Fig. 1: (a) Orthogonal projection of a triaxial ellipsoid K onto a plane normal to a hitting pivotal plane $L^3_{2[P]}$. (b) Perspective view of the elliptical pivotal section $C_u := K \cap L^3_{2[P]}$ of centre C , and of its orthogonal projection, of centre C' , onto the equatorial plane Ox_1x_2 . (c) Elliptical pivotal section with the three random variables involved in the unbiased estimator $\hat{S}$ of the surface area of the ellipsoid, see Eq. 7.

- Direction cosines of a unit direction vector $u \in \mathbb{S}^2_+$:

$$u := (u_1, u_2, u_3), \quad \left(\sum_{i=1}^3 u_i^2 = 1 \right). \quad (15)$$

In spherical polar coordinates,

$$u(\phi, \theta) := (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta), \quad (\phi \in [0, 2\pi], \theta \in [0, \pi/2]). \quad (16)$$

- Half orthogonal linear projection of K onto an axis of direction u , see Fig. 1a:

$$H(a, u) = \left(\sum_{i=1}^3 a_i^2 u_i^2 \right)^{1/2}, \quad (17)$$

(Appendix A.2). Actually, $H(a, u)$ is the support function of K , also known as half the caliper length of K along the direction u .

- Pivotal plane $L^3_{2[P]}$ through $P(x_0)$ with normal direction u :

$$L^3_{2[P]} := \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 : \sum_{i=1}^3 x_i u_i = p \right\}, \quad (18)$$

$$p \in (-H(a, u), H(a, u)),$$

where

$$p := p(x_0, u) = \sum_{i=1}^3 x_{0i} u_i \quad (19)$$

is the signed distance of $L^3_{2[P]}$ from the origin O , see Fig. 1a.

- The pivotal plane is isotropically oriented, which means that $u(\phi, \theta) \sim \text{UR}(\mathbb{S}^2_+)$. The joint probability element of (ϕ, θ) is,

$$\mathbb{P}(d\phi, d\theta) = \frac{d\phi}{2\pi} \cdot \sin \theta d\theta. \quad (20)$$

PARAMETRIZATION OF $\hat{S}$, AND COMPUTATION OF $\text{Var}(\hat{S})$

Further definitions and notation:

- Major and minor principal semiaxes of the section ellipse $C_u := K \cap L^3_{2[P]}$:

$$M := M(a, x_0, u), m := m(a, x_0, u), \quad (21)$$

respectively, see Fig. 1b,c.

- Major and minor principal semiaxes of a central ellipse $K \cap L^3_{2[0]}$ normal to u :

$$M_0 := M(a, 0, u), m_0 := m(a, 0, u), \quad (22)$$

respectively.

- Distance of the pivotal point $P(x_0)$ from the centre C of the section ellipse C_u :

$$r := r(a, x_0, u), \quad (23)$$

see Fig. 1b, c.

Now Eq. 7 may be rewritten as follows,

$$\hat{S} := \hat{S}(a, x_0, u) = 2\pi (M^2(a, x_0, u) + m^2(a, x_0, u) + r^2(a, x_0, u)), \quad (24)$$

and the unbiasedness of $\widehat{S}$ means that its expectation with respect to the uniform probability element given by Eq. 20 is equal to S , namely,

$$\begin{aligned} \mathbb{E} \left\{ \widehat{S}(a, x_0, u) \right\} \\ = \frac{1}{2\pi} \int_0^{\pi/2} \sin \theta \, d\theta \int_0^{2\pi} \widehat{S}(a, x_0, u(\phi, \theta)) \, d\phi \quad (25) \\ = S. \end{aligned}$$

The explicit formula sought for $\widehat{S}(a, x_0, u)$ is given by Proposition 1 below, and that of its variance by Corollary 1.

Lemma 1.

$$M_0^2 + m_0^2 = \sum_{1 \leq i < j \leq 3} a_i^2 a_j^2 (u_i^2 + u_j^2) / H^2(a, u), \quad (26)$$

(Appendix A.3).

Lemma 2.

$$M^2 + m^2 = (M_0^2 + m_0^2) \left(1 - \frac{p^2(x_0, u)}{H^2(a, u)} \right), \quad (27)$$

(Appendix A.4).

Lemma 3. Let $c := (c_1, c_2, c_3)$ denote the Cartesian coordinates of the centre C of the pivotal ellipse section $C_u := K \cap L_{2[P]}^3$. Then,

$$r^2 = \sum_{i=1}^3 (x_{0i} - c_i)^2 = \sum_{i=1}^3 \left(x_{0i} - \frac{a_i^2 u_i p(x_0, u)}{H^2(a, u)} \right)^2, \quad (28)$$

(Appendix A.5).

Proposition 1. Consider an ellipsoid $K \subset \mathbb{R}^3$ of principal semiaxes a , plus a given pivotal point P of Cartesian coordinates $x_0 \in K^\circ$, and a pivotal plane $L_{2[P]}^3$ through P with normal direction u , uniformly distributed in $\mathbb{S}_+^2$. Then,

$$\begin{aligned} \frac{1}{2\pi} \widehat{S}(a, x_0, u) &= \frac{\sum_{1 \leq i < j \leq 3} a_i^2 a_j^2 (u_i^2 + u_j^2)}{\sum_{i=1}^3 a_i^2 u_i^2} \\ &\times \left(1 - \frac{(\sum_{i=1}^3 x_{0i} u_i)^2}{\sum_{i=1}^3 a_i^2 u_i^2} \right) \\ &+ \sum_{i=1}^3 \left(x_{0i} - \frac{a_i^2 u_i \sum_{j=1}^3 x_{0j} u_j}{\sum_{j=1}^3 a_j^2 u_j^2} \right)^2 \end{aligned} \quad (29)$$

is an UE of the surface area S of K

Proof. It suffices to incorporate Eq. 26, Eq. 27 and Eq. 28 into the rhs of Eq. 24, bearing Eq. 17 and Eq. 19 in mind.

Corollary 1. The variance of the preceding estimator is,

$$\begin{aligned} \text{Var} \left(\widehat{S}(a, x_0, u) \right) \\ = \frac{1}{2\pi} \int_0^{\pi/2} \sin \theta \, d\theta \int_0^{2\pi} (\widehat{S}(a, x_0, u(\phi, \theta)))^2 \, d\phi \quad (30) \\ - S^2, \end{aligned}$$

where the direction cosines involved in Eq. 29 have been expressed in spherical polar coordinates, (Eq. 16).

Proof. Eq. 30 follows by the definition of the variance and by Eq. 25.

Corollary 2. The surface area $S(a_1, a_2, a_3)$ of an ellipsoid, see Eq. 10, may be expressed as follows,

$$\begin{aligned} S(a_1, a_2, a_3) \\ = \int_{u \in \mathbb{S}_+^2} \left(\sum_{1 \leq i < j \leq 3} a_i^2 a_j^2 (u_i^2 + u_j^2) \right) / \left(\sum_{i=1}^3 a_i^2 u_i^2 \right) \, du \\ = \int_{-1}^1 du_1 \int_0^{\sqrt{1-u_1^2}} \frac{\sum_{1 \leq i < j \leq 3} a_i^2 a_j^2 (u_i^2 + u_j^2)}{\sqrt{1-u_1^2-u_2^2} \sum_{i=1}^3 a_i^2 u_i^2} \, du_2 \quad (31) \end{aligned}$$

with the substitution $u_3^2 = 1 - u_1^2 - u_2^2$ in the latter integrand.

Proof. The estimator $\widehat{S}(a, x_0, u)$, see Eq. 29, is unbiased for any choice of the pivotal point $P(x_0) \in K^\circ$. For the particular choice $x_0 = (0, 0, 0)$, the expected value of $\widehat{S}(a, 0, u)$ with respect to the uniform probability element

$$\begin{aligned} \mathbb{P}(du) &= \frac{du}{2\pi} = \frac{du_1 \, du_2}{2\pi \sqrt{1-u_1^2-u_2^2}} \quad (32) \\ &\left(-1 \leq u_1 \leq 1, 0 \leq u_2 \leq \sqrt{1-u_1^2} \right) \end{aligned}$$

must therefore be equal to the rhs of Eq. 10.

As an exercise, a direct verification of Eq. 31, using spherical polar coordinates, is given in Appendix A.6.

Corollary 3. If K is the unit sphere, namely if $a_1 = a_2 = a_3 = 1$, then by symmetry we may take $x_{01} = x_{02} = 0$ and $u_1 = \sin \theta, u_2 = 0, u_3 = \cos \theta$, whereby Eq. 29 becomes,

$$\widehat{S}(x_{03}, \theta) = 2\pi (2 - 2x_{03}^2 + 3x_{03}^2 \sin^2 \theta). \quad (33)$$

Then, the application of Eq. 25 and Eq. 30 yield, respectively,

$$\mathbb{E}\{\hat{S}(x_{03}, \theta)\} = 4\pi = S$$

$$\text{CV}^2\{\hat{S}(x_{03}, \theta)\} = \frac{\text{Var}\{\hat{S}(x_{03}, \theta)\}}{S^2} = \frac{x_{03}^4}{5}. \quad (34)$$

For the unit sphere, and for the estimator of S given by Eq. 5,

$$\text{CV}^2\{\hat{S}(u, \omega)\} = \frac{4x_{03}^2}{3}, \quad (35)$$

(Cruz-Orive, 2008, Eq. 6), which is larger than $x_{03}^4/5$ for all $x_{03} \in (0, 1)$, as expected from Eq. 8.

HEAT MAPS OF CV ($\hat{S}$) FOR A GIVEN ELLIPSOID

While the unbiasedness identity $\mathbb{E}(\hat{S}) = S$ does not depend on the location of the pivotal point $x_0 \in K^\circ$, the variance of $\hat{S}$ does. For any arbitrary pivotal point in the interior of a given ellipsoid, the numerical value of the double integral in the rhs of Eq. 30 can be evaluated quickly with the aid of software packages such as R, (<http://www.r-project.org>), or Mathematica, (Wolfram Research, Inc.).

To appreciate the behaviour of $\text{CV}(\hat{S})$ as the location of the pivotal point varies, it suffices to adopt an arbitrary triaxial ellipsoid - here, one of principal semiaxes $a = (1, 2, 3)$ was chosen for that purpose, see Fig. 2. The application of Eq. 10 yields $S = 48.8821 \dots$ for this ellipsoid. As expected, numerical integration of the rhs of Eq. 25 yields the same value for arbitrary choices of x_0 satisfying Eq. 14.

However, $\text{Var}(\hat{S})$ depends on the choice of x_0 . For instance, for $x_0 = (0.3, 0.0, 1.7)$, $(0.0, 0.4, 2.4)$, and $(0.5, 0.7, 0.0)$, say, the corresponding values of $\text{CV}(\hat{S}) = \sqrt{\text{Var}(\hat{S})/S}$ are $0.44 \dots$, $0.65 \dots$, $0.18 \dots$. For numerical computation, the vector u in Eq. 29 was expressed in spherical polar coordinates.

The results given in the present section were computed with the aid of the software package R4.4.2 (<http://www.r-project.org>). The computation of the elliptic integrals involved in Eq. 10 requires calling `library(Carlson)`.

The heat maps displayed in Fig. 2 (upper panels) correspond to the cases in which the pivotal point belongs to each of the intersections of the ellipsoid with the planes $x_2 = 0$, $x_1 = 0$, and $x_3 = 0$, respectively, namely the three principal ellipses. Within each principal ellipse, pivotal points were generated constituting a square grid of size 0.01. For

instance, the principal ellipse in the plane $x_2 = 0$ was enclosed in the rectangle $[-1, 1] \times [-3, 3]$, which contained 201×601 grid points. As a check, the proportion of pivotal points within the ellipse section, namely satisfying the condition $x_1^2 + x_3^2/9 < 1$, was of $94197/(200 \times 600)$, which is approximately equal to the expected ratio $\pi/4$ of the ellipse area to the rectangle area. Based on the colour intensity of the maps, it appears that the larger values of $\text{CV}(\hat{S})$ correspond to pivotal points in peripheral regions of the ellipsoid. The lower panels in Fig. 2 suggest that the variation of $\text{CV}(\hat{S})$ as a function of the distance from the ellipsoid centre of x_0 along an axis oscillates, hence it will not be monotonic in general.

The R program used was supplied by Claude.ai. It required calling `library(ggplot2)`, `library(scales)`, and `library(viridis)`. The codes of the programs used in this paper (under Windows 10) are available from the author on request.

FINAL COMMENTS

APPLICABILITY

While the estimator $\hat{S}$ can be computed by Eq. 7 from measurements made in any pivotal ellipse section of an arbitrary ellipsoid with unknown principal semiaxes lengths, $\text{Var}(\hat{S})$ cannot be estimated from such measurements. The numerical computation of the exact $\text{Var}(\hat{S})$ via Eq. 30 is not a problem, but it requires the knowledge of the mentioned lengths, and of the coordinates of the pivotal point.

In an (uncommon) practical case involving a population of arbitrary ellipsoidal particles containing a sole, prominent pivotal point (e.g., a 'nucleolus' in some biological cells), a ratio unbiased estimator of the ordinary population mean $\mathbb{E}_N(S)$ of the individual ellipsoid surface areas is,

$$\bar{s}_N = \frac{1}{Q^-} \sum_{i=1}^{Q^-} s_i = \frac{2\pi}{Q^-} \sum_{i=1}^{Q^-} (M_i^2 + m_i^2 + r_i^2), \quad (36)$$

which is an adaptation to ellipsoids of the first Eq. (4.2) from Cruz-Orive (2005), or of Eq. (4.24.2) from Cruz-Orive (2024). Thus, Q^- is the total number of ellipsoids sampled with isotropic uniform random (IUR) optical disectors, say, using the nucleoli as sampling units. A sweeping optical section meeting the i th nucleolus for the first time is thereby a pivotal section, and the corresponding lengths (M_i, m_i, r_i) can be measured in the corresponding ellipse section by adopting the sampled nucleolus as the pivotal point. If a particle contains several nucleoli, then one of them is chosen uniformly at random.

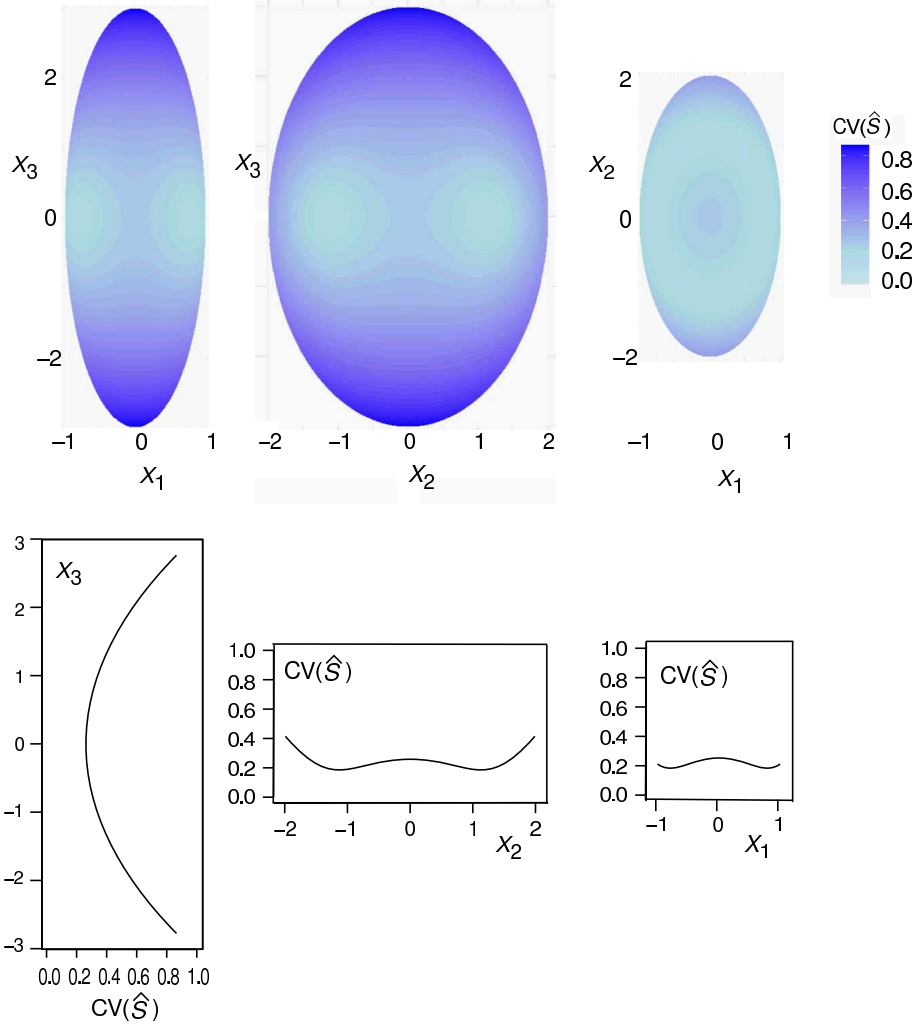


Fig. 2: Above: Heat maps of the coefficient of variation $CV(\hat{S})$ of the estimator $\hat{S}$ computed via Eq. 30, for a triaxial ellipsoid of principal semi-axes (1, 2, 3). From left to right, intersections of the ellipsoid with the planes $x_2 = 0$, $x_1 = 0$, and $x_3 = 0$, respectively. Below: Corresponding plots of $CV(\hat{S})$ as the pivotal point lies along each of the three principal axes of the ellipsoid.

If the ellipsoidal particles do not contain nucleoli, then to estimate $\mathbb{E}_N(S)$ one can resort to a classical ratio design, e.g. Cruz-Orive (2024), Section 2.31. An alternative is to consider instead the volume weighted mean $\mathbb{E}_V(S) := \mathbb{E}_N(VS)/\mathbb{E}_N(V)$, which does not require disectors. Superimpose a test system of points on a planar IUR section through the population, and let $p_i \geq 1$, ($i = 1, 2, \dots, n$), denote the total number of test points hitting the i th hit ellipse section (n , which does not enter in the final estimator, is the number of hit ellipse sections). Actually, p_i is the number of times the i th ellipsoid volume is sampled, hence the volume weighting effect. Each hitting point is effectively a pivotal point, and therefore the invariator estimator of the surface area of the i th hit ellipsoid, based on the p_i test points hitting it, is,

$$s_i = \frac{2\pi}{p_i} \sum_{j=1}^{p_i} (M_{ij}^2 + m_{ij}^2 + r_{ij}^2). \quad (37)$$

Set $P = \sum_{i=1}^n p_i$, the total number of test points hitting ellipse sections in the IUR section plane. Then, a ratio unbiased estimator of $\mathbb{E}_V(S)$ is,

$$\bar{s}_V = \frac{1}{P} \sum_{i=1}^n p_i s_i = \frac{2\pi}{P} \sum_{i=1}^n \sum_{j=1}^{p_i} (M_{ij}^2 + m_{ij}^2 + r_{ij}^2). \quad (38)$$

Renumber the entire set of P measurement triplets $\{(M_{ij}, m_{ij}, r_{ij})\}$ as $\{(M_i, m_i, r_i), i = 1, 2, \dots, P\}$. Then,

$$\bar{s}_V = \frac{2\pi}{P} \sum_{i=1}^P (M_i^2 + m_i^2 + r_i^2). \quad (39)$$

The preceding estimator is an adaptation to ellipsoids

of the first Eq. 5.3 from Cruz-Orive (2005), or of Eq. 4.25.1 from Cruz-Orive (2024).

GENERALIZATION

Computable formulae exist for the surface area S_d of the d -dimensional ellipsoid, see e.g. Rivin (2007), who generalizes formulae of Moran (1982) for $d = 3$. For $d > 3$, however, no closed expression is known in terms of elementary, or of classical special functions like elliptic integrals. The question arises of whether Eq. 31 could be generalized. Eq. 21 from Gual-Arnau et al. (2010) generalizes Eq. 2 for odd d only, namely for $d = 2r + 1, r = 0, 1, \dots$, and the pivotal plane has to be replaced with a subspace $L_{r+1[0]}^d$. In the invariator context, Gual-Arnau (2023) considers a cylinder probe of radius R (instead of a weighted test line $L_{1(z)}^2$) within a pivotal slab of thickness $2R$. The latter author does not consider the special case of convex bodies. It seems unlikely that Eq. 31 can be generalized.

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APPENDIX

A.1. DERIVATION OF EQ. 6 FOR THE FLOWER AREA OF AN ELLIPSE.

For a given pivotal ellipse section C_u , Eq. 4, with $h_P(\omega) := h(u, \omega)$, is equivalent to

$$A(H_{C_u}) = \frac{1}{2} \int_0^{2\pi} h_P^2(\omega) d\omega. \quad (40)$$

The support function $h_P(\omega)$ of C_u , with origin P , is just the radius vector of H_{C_u} with centre P . From Fig. 3 it is seen that $h_P(\omega)$ is equal to half the caliper length $H(\omega)$ of C_u , minus the orthogonal projected length of the vector $\overline{CP}$ onto of the support line of $H(\omega)$. Let (x_0, y_0)

denote the Cartesian coordinates of P with respect to the ellipse centre C . Then,

$$\begin{aligned} h_P(\omega) &= H(\omega) - (x_0, y_0) \cdot (\cos \omega, \sin \omega) \\ &= (M^2 \cos^2 \omega + m^2 \sin^2 \omega)^{1/2} \\ &\quad - (x_0 \cos \omega + y_0 \sin \omega). \end{aligned} \quad (41)$$

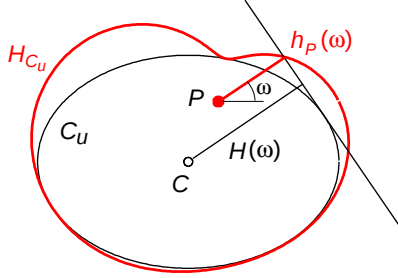


Fig. 3: Pivotal ellipse C_u , see also Fig. 1, incorporating the corresponding support set or 'flower' H_{C_u} with respect to a pivotal point P .

The expression of $H(\omega)$ is a particular case of Eq. 52 for $d = 2$. Now Eq. 40 yields,

$$\begin{aligned} A(H_{C_u}) &= \frac{1}{2} \int_0^{2\pi} h_P^2(\omega) d\omega \\ &= \frac{1}{2} \int_0^{2\pi} (M^2 \cos^2 \omega + m^2 \sin^2 \omega \\ &\quad + x_0^2 \cos^2 \omega + y_0^2 \sin^2 \omega) d\omega, \end{aligned} \quad (42)$$

because the integrals of the terms involving the first powers of $\sin \omega$, or $\cos \omega$, vanish. Thus,

$$\begin{aligned} A(H_{C_u}) &= \frac{\pi}{2} (M^2 + m^2 + x_0^2 + y_0^2) \\ &= \frac{\pi}{2} (M^2 + m^2 + r^2). \end{aligned} \quad (43)$$

A.2. DERIVATION OF EQ. 17 FOR $H(a, u)$

To make the exposition self-contained, this, and Appendix A.4, are adapted from the Appendix in Cruz-Orive (1985). Here it is convenient to use matrix calculus. No difficulty is added by working in $\mathbb{R}^d$. Consider a d -ball $B_d(h)$ of radius h , centred at O . Referred to an orthogonal frame $Oy_1y_2 \cdots y_d$, the equation of the ball is,

$$\begin{aligned} y'By &\leq 0, \\ y &= (y_1, y_2, \dots, y_d, 1)', \\ B &= \text{diag}(h^{-2}, h^{-2}, \dots, {}^d h^{-2}, -1). \end{aligned} \quad (44)$$

The equation of an hyperplane tangent to the ball is,

$$\begin{aligned} t'(h)y &= 0, \\ t(h) &= (t_1, t_2, \dots, t_d, -h)', \\ (t_1^2 + t_2^2 + \dots + t_d^2 &= 1). \end{aligned} \quad (45)$$

The following unimodular central affinity,

$$\begin{aligned} x &= Gy, \\ x &= (x_1, x_2, \dots, x_d, 1)' \\ G &= \text{diag}(a_1/h, a_2/h, \dots, a_d/h, 1), \\ h &= (a_1 a_2 \cdots a_d)^{1/d}, \end{aligned} \quad (46)$$

transforms the ball, Eq. 44, into

$$\begin{aligned} x'Ax &\leq 0, \\ A &= G^{-1}BG^{-1} \\ &= \text{diag}(a_1^{-2}, a_2^{-2}, \dots, a_d^{-2}, -1), \end{aligned} \quad (47)$$

namely a d -ellipsoid $K_d(a)$, $a := (a_1, a_2, \dots, a_d)$, and it also transforms the tangent hyperplane given by Eq. 45 into the hyperplane

$$t'(h)G^{-1}x = 0, \text{ namely } \sum_{i=1}^d (t_i/a_i)x_i = 1, \quad (48)$$

which is tangent to the ellipsoid. If the coefficients $u_1, u_2, \dots, u_d$, say, of $x_1, x_2, \dots, x_d$, respectively, satisfied the identity $u_1^2 + u_2^2 + \dots + u_d^2 = 1$, then the rhs of the resulting equation would be the distance of the tangent hyperplane from O , namely the support function $H(a, u)$ of the ellipsoid with respect to O . To achieve this, set

$$\begin{aligned} u_i &= (t_i/a_i) \left(\sum_{j=1}^d t_j^2/a_j^2 \right)^{-1/2}, \\ (u_1^2 + u_2^2 + \dots + u_d^2 &= 1), \end{aligned} \quad (49)$$

whereby,

$$\sum_{i=1}^d a_i^2 u_i^2 \sum_{i=1}^d t_j^2/a_j^2 = \sum_{i=1}^d t_i^2 = 1, \quad (50)$$

so that the transformed hyperplane becomes,

$$\sum_{i=1}^d x_i u_i \left(\sum_{j=1}^d t_j^2/a_j^2 \right)^{1/2} = 1, \quad (51)$$

or,

$$\sum_{i=1}^d x_i u_i = \left(\sum_{i=1}^d a_i^2 u_i^2 \right)^{1/2} = H(a, u). \quad (52)$$

For $d = 3$, $H(a, u)$ becomes Eq. 17.

A.3. PROOF OF LEMMA 1

The equation of the boundary ∂K of the ellipsoid K given by Eq. 9 is $f_1(x_1, x_2, x_3) = 1$, where

$$f_1 := f_1(x_1, x_2, x_3) = x_1^2/a_1^2 + x_2^2/a_2^2 + x_3^2/a_3^2. \quad (53)$$

On the other hand, the equation of a plane $L_{2[0]}^3$ through the origin is $f_2(x_1, x_2, x_3) = 0$, where

$$f_2 := f_2(x_1, x_2, x_3) = x_1 u_1 + x_2 u_2 + x_3 u_3, \\ (u_1^2 + u_2^2 + u_3^2 = 1). \quad (54)$$

The square semiaxes m_0^2, M_0^2 of the central section $K \cap L_{2[0]}^3$, see Eq. 22, are the minimum and the maximum, respectively, of the square distance of a point of the boundary ∂K from O , namely of the function

$$R^2 := R^2(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2, \quad (55)$$

with the restrictions given by $f_1 = 1$ and $f_2 = 0$. By the Lagrange multipliers method, two parameters λ_1, λ_2 are introduced, and the function to minimize is,

$$f = R^2 + \lambda_1 (f_1 - 1) + 2\lambda_2 f_2. \quad (56)$$

The procedure leads to the following three equations,

$$\left\{ \frac{\partial f}{\partial x_i} = 0, i = 1, 2, 3 \right\}. \quad (57)$$

Multiplying the i th equation with x_i , adding up from $i = 1$ to $i = 3$, and making use of the definitions of R^2, f_1, f_2 , one obtains $\lambda_1 = -R^2$. Now, Eq. 57 yields,

$$\left\{ x_i = \frac{\lambda_2 a_i^2 u_i}{R^2 - a_i^2}, i = 1, 2, 3 \right\}, \quad (58)$$

which, substituted into the equation $f_2(x_1, x_2, x_3) = 0$, yields the following quadratic equation in the unknown R^2 ,

$$H^2 R^4 - H^2 \mu(a, u) R^2 + (a_1 a_2 a_3)^2 = 0 \quad (59)$$

where $H := H(a, u)$, see Eq. 17, whereas the factor $\mu(a, u)$ has in fact the expression in the rhs of Eq. 26. But $\mu(a, u)$ is precisely the sum of the two roots of R^2 , namely $M_0^2 + m_0^2$. As a check, the area of the central ellipsoid is,

$$A(K \cap L_{2[0]}^3) = \pi M_0 m_0 = \frac{\pi a_1 a_2 a_3}{H} = \frac{3V(K)}{4H}, \quad (60)$$

which is a special case of Eq. 67, see below, for $d = 3$. The foregoing proof is adapted from an exercise found in Rey-Pastor (1967, pp. 316-7), where the purpose was to find the expressions of m_0 and M_0 .

A.4. PROOF OF LEMMA 2

Consider a d -ellipsoid $K_d(a)$, see Eq. 47, transected by an hyperplane $L_{d-1}^d(p, u)$ of equation

$$u'(p)x = 0, \\ u(p) = (u_1, u_2, \dots, u_d, -p)', \quad (61) \\ (u_1^2 + u_2^2 + \dots + u_d^2 = 1), |p| \leq H(a, u).$$

Set,

$$V_{d-1}(p, u) := \text{Volume} \left(K_d(a) \cap L_{d-1}^d(p, u) \right). \quad (62)$$

Under the transform $x = Gy$, see Eq. 46, the d -ellipsoid becomes a d -ball $B_d(h)$, Eq. 44, whereas the hyperplane transecting the ball becomes

$$u'(p)Gy = 0, \text{ namely } \sum_{i=1}^d a_i y_i u_i = ph. \quad (63)$$

Analogously as in Appendix A.2, set

$$t_i = a_i u_i \left(\sum_{i=1}^d a_i^2 u_i^2 \right)^{-1/2} = a_i u_i / H, \quad (64)$$

where $H := H(a, u)$, for short. Then, Eq. 63 becomes

$$\sum_{i=1}^d y_i t_i = ph/H, \quad (t_1^2 + t_2^2 + \dots + t_d^2 = 1), \quad (65)$$

and therefore $B_d(h) \cap L_{d-1}^d(ph/H, t)$ is a $(d-1)$ -ball of radius $(h^2 - p^2 h^2 / H^2)^{1/2}$ and volume $c(d-1)h^{d-1}(1 - p^2/H^2)^{(d-1)/2}$, where $c(d) = \pi^{d/2}/\Gamma(d/2 + 1)$ is the volume of the unit d -ball. Because G is unimodular and it transforms a vector of length H into one of length h , it follows that the ratio of the ellipsoid transect volume to the ball transect volume must be equal to h/H , whereby,

$$V_{d-1}(p, u) = H^{-1} c(d-1) h^d (1 - p^2/H^2)^{(d-1)/2} \\ = V_{d-1}(0, u) (1 - p^2/H^2)^{(d-1)/2}, \quad (66)$$

(Cruz-Orive, 1985, Jensen & Møller, 1986), where

$$V_{d-1}(0, u) = c(d-1) h^d / H \\ = c(d-1) a_1 a_2 \dots a_d / H \quad (67)$$

is the volume of the central transect of the ellipsoid. For $d = 3$, Eq. 66 yields,

$$\pi M m = \pi M_0 m_0 (1 - p^2/H^2). \quad (68)$$

Parallel ellipsoid sections are similar ellipses. In fact, under the transform G , which is linear, parallel ball

sections, namely disks, will become similar ellipses. Thus,

$$M/M_0 = m/m_0, \quad (69)$$

whereby Eq. 68 yields,

$$\begin{aligned} M^2 &= M_0^2 (1 - p^2/H^2), \\ m^2 &= m_0^2 (1 - p^2/H^2), \end{aligned} \quad (70)$$

and therefore,

$$M^2 + m^2 = (M_0^2 + m_0^2) (1 - p^2/H^2), \quad (71)$$

which is Eq. 27.

Results equivalent to Eq. 70 are well known for spheroids, e.g. Cruz-Orive (1976), but not readily available for triaxial ellipsoids in that simple form. Recall that M_0^2, m_0^2 are the roots of Eq. 59.

A.5. PROOF OF LEMMA 3

The problem is to parametrize the centre $C(c_1, c_2, c_3)$ of the pivotal ellipse section $C_u := K \cap L_{2[P]}^3$. The orthogonal projection of C_u onto the plane $x_3 = 0$ is an ellipse C'_u , say, of centre $C'(c_1, c_2)$, see Fig. 1b, whereas

$$c_3 = (p(u, x_0) - u_1 c_1 - u_2 c_2) / u_3, \quad (72)$$

because $C(c_1, c_2, c_3) \in L_{2[P]}^3$. Thus, the problem is reduced to finding $C'(c_1, c_2)$. The equation of the ellipse C'_u is obtained by eliminating x_3 from Eq. 9 and Eq. 18. In homogeneous coordinates the result is,

$$\begin{aligned} f(x_1, x_2, t) &:= a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2 \\ &+ 2a_{13}x_1t + 2a_{23}x_2t + a_{33}t^2 = 0, \end{aligned} \quad (73)$$

where,

$$\begin{aligned} a_{11} &= u_1^2 + u_3^2 a_3^2 / a_1^2, \\ a_{12} &= a_{21} = u_1 u_2, \\ a_{13} &= a_{31} = -p u_1, \\ a_{22} &= u_2^2 + u_3^2 a_3^2 / a_2^2, \\ a_{23} &= a_{32} = -p u_2, \\ a_{33} &= p^2 - u_3^2 a_3^2. \end{aligned} \quad (74)$$

The equation of the polar line of C'_u with respect to an exterior pole (x'_1, x'_2, t') is

$$g(x_1, x_2, t, x'_1, x'_2, t') := x'_1 \frac{\partial f}{\partial x_1} + x'_2 \frac{\partial f}{\partial x_2} + t' \frac{\partial f}{\partial t} = 0. \quad (75)$$

The ellipse centre is the point of intersection (c_1, c_2) of the two polar lines corresponding to the ideal poles $(1, 0, 0)$ and $(0, 1, 0)$. The corresponding equations are,

$$\begin{cases} g(c_1, c_2, 1, 1, 0, 0) = 0 \\ g(c_1, c_2, 1, 0, 1, 0) = 0 \end{cases} \quad (76)$$

namely,

$$\begin{cases} a_{11}c_1 + a_{12}c_2 + a_{13} = 0 \\ a_{12}c_1 + a_{22}c_2 + a_{23} = 0 \end{cases}$$

Solving for (c_1, c_2) , replacing the $\{a_{ij}\}$ with their expressions in Eq. 74, and using Eq. 72, Eq. 28 is obtained.

A.6. SYMBOLIC COMPUTATION OF THE DOUBLE INTEGRAL IN EQ. 31

The ensuing verification is strictly not necessary because, by Corollary 2, Eq. 31 must hold. Nonetheless, the integral is non trivial, and its direct computation may suit readers willing to circumvent the underlying stereological theory of the invariator design.

To avoid singularities, it is convenient to use spherical polar coordinates, see Eq. 16, whereby the identity to be verified is,

$$\int_0^{\pi/2} \sin \theta \, d\theta \int_0^{2\pi} f(\phi, \theta) d\phi = S(a_1, a_2, a_3) \quad (78)$$

where,

$$\begin{aligned} f(\phi, \theta) &= \sum_{i=1}^3 f_i(\phi, \theta), \\ f_1(\phi, \theta) &= a_1^2 a_2^2 \sin^2 \theta / H^2(\phi, \theta), \\ f_2(\phi, \theta) &= a_1^2 a_3^2 (1 - \sin^2 \theta \sin^2 \phi) / H^2(\phi, \theta), \\ f_3(\phi, \theta) &= a_2^2 a_3^2 (1 - \sin^2 \theta \cos^2 \phi) / H^2(\phi, \theta), \\ H^2(\phi, \theta) &= a_1^2 \sin^2 \theta \cos^2 \phi + a_2^2 \sin^2 \theta \sin^2 \phi \\ &+ a_3^2 \cos^2 \theta \end{aligned} \quad (79)$$

Mathematica 7, (Wolfram Research, Inc.) was not effective in checking Eq. 78 directly. However, integration was possible by treating each of the preceding three functions separately. Furthermore, integration was first carried out with respect to ϕ , the corresponding result was simplified, multiplied with $\sin \theta$, and finally integrated with respect to θ . For instance,

$$\begin{aligned} g_1(\theta) &:= \int_0^{2\pi} f_1(\phi, \theta) d\phi \\ &= \frac{4\pi a_1^2 a_2^2 \sin^2 \theta}{\sqrt{\prod_{i=1}^2 (a_i^2 + a_3^2 + (a_3^2 - a_i^2) \cos(2\theta))}}, \end{aligned} \quad (80)$$

and then,

$$G_1(a_1, a_2, a_3) := \int_0^{\pi/2} g_1(\theta) \sin \theta d\theta \\ = \frac{2\pi i a_1^2 a_2 (a_2^2 E(i\alpha, q) - a_3^2 F(i\alpha, q))}{(a_3^2 - a_2^2) \sqrt{a_3^2 - a_1^2}}, \quad (81)$$

where

$$\alpha = \operatorname{csch}^{-1} \left(a_1 / \sqrt{a_3^2 - a_1^2} \right), \quad (82) \\ q^2 = (a_3^2/a_2^2 - 1) / (a_3^2/a_1^2 - 1).$$

In Eq. 81, the two elliptic integrals of imaginary amplitude are transformed into the ones with real amplitude with the aid of Eq. 17.4.8 and Eq. 17.4.9 from Abramowitz and Stegun (1965), namely,

$$E(i\alpha, q) = -iE(\varphi, k) + iF(\varphi, k) + i\sqrt{a_3^2 - a_1^2}/a_2 \\ F(i\alpha, q) = iF(\varphi, k) \quad (83)$$

where

$$\cos \varphi = \cos(\tan^{-1}(\sinh(\alpha))) = a_1/a_3 \quad (84) \\ k^2 = 1 - q^2 = (1 - a_1^2/a_2^2) / (1 - a_1^2/a_3^2)$$

in agreement with Eq. 11. Thus, Eq. 81 becomes,

$$G_1(a_1, a_2, a_3) = \frac{2\pi a_1^2 a_2^2}{a_2^2 - a_3^2} \\ + \frac{2\pi a_1^2 a_2 (a_2^2 E(\varphi, k) + (a_3^2 - a_2^2) F(\varphi, k))}{(a_3^2 - a_2^2) \sqrt{a_3^2 - a_1^2}}. \quad (85)$$

Similarly, the double integrals corresponding to f_2, f_3 become,

$$G_2(a_1, a_2, a_3) = \frac{2\pi a_1^2 a_3^2}{a_3^2 - a_2^2} \\ - \frac{2\pi a_1^2 a_2 a_3^2 ((a_3^2 - a_1^2) E(\varphi, k) - (a_3^2 - a_2^2) F(\varphi, k))}{(a_2^2 - a_1^2) (a_3^2 - a_2^2) \sqrt{a_3^2 - a_1^2}}, \\ G_3(a_1, a_2, a_3) = \frac{2\pi a_2 a_3^2 (a_2^2 E(\varphi, k) - a_1^2 F(\varphi, k))}{(a_2^2 - a_1^2) \sqrt{a_3^2 - a_1^2}}, \quad (86)$$

respectively. It is readily verified that

$$\int_0^{\pi/2} \sin \theta d\theta \int_0^{2\pi} f(\phi, \theta) d\phi \\ = \sum_{i=1}^3 G_i(a_1, a_2, a_3) = 2\pi a_1^2 \quad (87) \\ + \frac{2\pi a_2}{\sqrt{a_3^2 - a_1^2}} ((a_3^2 - a_1^2) E(\varphi, k) + a_1^2 F(\varphi, k))$$

which coincides with the rhs of Eq. 10, as required.