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RECTILINEAR AND BROWNIAN MOTION FROM A RANDOM POINT IN A CONVEX REGION

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ABSTRACT

A particle is projected from a point *P* in a subset *E* of a convex region *H* to a point *Q* in a uniformly random direction. The probability that *Q* lies in the interior of *H* at time *t* is obtained for two types of motion of the particle, rectilinear (i.e. straight-line) and Brownian. In the case of rectilinear motion, the first passage time through the boundary of *H* is considered. Results are obtained in terms of the generalized overlap function for embedded bodies.

Keywords: Brownian motion, convex body, geometric probability, random ray.

INTRODUCTION

Consider a particle moving from point P at time $t = 0$ to point Q at $t > 0$. P is taken to lie in a subset E of a compact convex body $H \subset R^n$. The set E may be nonconvex. It may be disconnected and may have lower dimensionality than H; indeed, E may consist of a single point (see Fig. 1). In the next section we relate the probability that Q lies in the interior of H to the generalized overlap function for embedded bodies introduced by Enns and Ehlers (1988). Thereafter we obtain this probability first for the case of motion along a straight path and then for the case of Brownian motion. Depending on its equations of motion, the particle may leave and re-enter the region H any number of times. We are concerned only with its presence or absence inside H at time t.

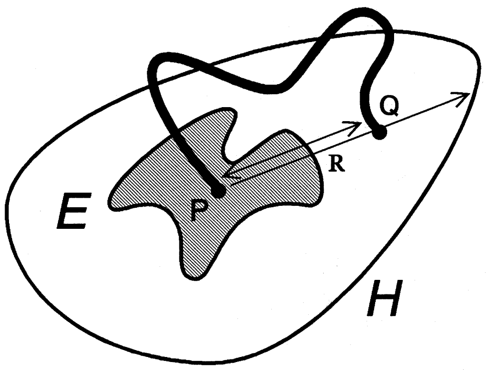

Fig. 1. *A particle moves from P to Q in time t.*

There is a considerable literature on escape processes and first exit times, especially as related to the example of Brownian motion; see, for example, Getoor (1979) or Wendel (1980). For more general books, see Gihman and Skorohod (1975) or Knight (1981).

GENERAL FORMULATION

Let $Q(t)$ denote the location of the particle at time $t \geq 0$. Let $P = Q(0)$ be a uniformly random starting point in E. Denote the distance between $Q(0)$ and $Q(t)$ by $X(t)$. Assuming that motion is isotropic, it is possible to relate the probability of finding $Q(t)$ inside H to the overlap function which is defined by

$$\Omega_{E,H}(x) \equiv \frac{E_\theta\left[vol\left(E(x,\theta) \cap H\right)\right]}{vol\left(E\right)}$$

where $E(x,\theta)$ is the translate of E in direction θ by a distance x, $vol(\bullet)$ denotes volume and $\mathbf{E}_\theta(\bullet)$ indicates direction average with θ uniformly random over all directions.

Let

$$h(t) = \mathbf{P}(Q(t) \in H)$$

and

$$F_X(r,t) = \mathbf{P}(X(t) \le r).$$

The following theorem relates $h(t)$ to the overlap function.

Theorem

$$h(t) = \int \Omega_{E,H}(r)\, dF_X(r,t).$$

Proof. Let $B(r,P)$ denote the n-ball of radius r with centre at P and let $\partial B(r,P)$ denote its surface. Then

$$\begin{aligned}\mathbf{P}(Q(t) \in H \mid X(t) = r; Q(0) = P) &= \frac{vol[H \cap \partial B(r,P)]}{vol[\partial B(r,P)]} \\ &= \frac{vol[H \cap \partial B(r,P)]}{nC_n r^{n-1}} \\ &= \frac{\phi(r,P)}{nC_n}\end{aligned}$$

where $C_n = \pi^{n/2} / \Gamma(\frac{n}{2}+1)$ is the volume of the unit n-ball and $\phi(r,P)$ is the solid angle subtended at P by $H \cap \partial B(r,P)$. Averaging with respect to net distance X and initial point P, one obtains the unconditional probability that $Q(t) \in H$:

$$\begin{aligned}h(t) &= \mathbf{P}(Q(t) \in H) \\ &= \mathbf{E}_P \mathbf{E}_X \mathbf{P}(Q(t) \in H \mid X(t); P) \\ &= \int_{r=0}^{\infty} \mathbf{E}_P \left[\frac{\phi(r,P)}{nC_n}\right] dF_X(r).\end{aligned}$$

But $\mathbf{E}_P(\phi(r,P)/nC_n) = \mathbf{\Omega}_{E,H}(r)$ (Enns and Ehlers, 1988). Therefore

$$h(t) = \int \Omega_{E,H}(r)\, dF_X(r,t) = \mathbf{E}_X\left[\Omega_{E,H}(X(t))\right], \quad (1)$$

relating $h(t)$ and the overlap function.

It was shown by Enns and Ehlers (1988) that $1 - \Omega_{E,H}(r)$ is the distribution function of the length R of a random ray generated by selecting a point in E and a direction, independently uniformly distributed and with terminal point in the boundary ∂H of H. We call this a ν-random ray. Writing $\Omega_{E,H}(r) = \mathbf{P}(R > r) = \bar{F}_R(r)$ results in

$$\begin{aligned}h(t) &= \int \bar{F}_R(r)\, dF_X(r,t) \\ &= \int F_X(r,t)\, dF_R(r) \\ &= \mathbf{E}_R(F_X(R,t)).\end{aligned}$$

Summarizing, we have

$$h(t) = \mathbf{E}_X\left[\Omega_{E,H}(X(t))\right] = \mathbf{E}_R(F_X(R,t)), \quad (2)$$

where the R-expectation is with respect to ν-measure.

We now turn to specific types of particle motion.

RECTILINEAR MOTION

For a particle undergoing straight-line motion with position function $u(t)$ along the directed half-line originating at P (direction equal to the direction of motion so that $u(t)$ is increasing), $X(t)$ has degenerate distribution

$$F_X(r,t) = I(r - u(t)),$$

where $I(\bullet)$ is the indicator function

$$I(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \le 0. \end{cases}$$

Substitution in Eq. 2 yields

$$h(t) = \Omega_{E,H}(u(t)).$$

With ∂H denoting the boundary of H, we define the random variable

$$T = \min\{t : Q(t) \in \partial H\}.$$

Thus T is the first exit time for a particle. It follows that $h(t) = \mathbf{P}(T > t) \equiv \bar{F}_T(t)$. Moments of T may then be obtained by evaluating

$$\mathbf{E}T^k = k\int_{t=0}^{\infty} t^{k-1} \bar{F}_T(t)\, dt = k\int_{t=0}^{\infty} t^{k-1} \Omega_{E,H}(u(t))\, dt. \quad (3)$$

For the case of motion at constant velocity, where $u(t) = vt$, Eq. 3 yields

$$\mathbf{E}T^k = v^{-k} k \int_{x=0}^{\infty} x^{k-1} \Omega_{E,H}(x)\, dx.$$

Obviously, this case corresponds to $T = R/v$, where R is the ν-random ray length from a uniformly random point in E to the boundary of H. A shape-independent moment for n-dimensional H is

$$\mathbf{E}T^n = \frac{vol(H)}{C_n v^n}.$$

Example. Enns and Ehlers (1988, 1993) give the overlap functions for the case of concentric balls. Let $H = B_n(b)$, $E = B_d(a)$, $a \le b$, and $d \le n = 3$.

This situation models the physically important cases of particles generated in a linear or circular region inside the 3-ball. We have $\Omega_{E,H}(x)=1$ for $x \le b-a$ and $\Omega_{E,H}(x)=0$ for $x \ge b+a$. In the interval $b-a<x<b+a$, the overlap functions for $d=1,2,3$ are:

$d=1$:

$$\Omega_{E,H}(x)=\frac{1}{8ax}\left[\left(b^2-a^2\right)+4ax+2bx-3x^2+2\left(x^2-b^2\right)\log\left|\frac{x-b}{a}\right|\right]$$

$d=2$:

$$\Omega_{E,H}(x)=\frac{1}{6a^2x}\left[3a\left(b^2-a^2-x^2\right)+3x\left(a^2+\left|x^2-b^2\right|\right)+2\left(a^3-\left|x^3-b^3\right|\right)\right]$$

$d=3$:

$$\Omega_{E,H}(x)=\frac{1}{16a^3x}\left[-3\left(b^2-a^2\right)^2+8\left(a^3+b^3\right)x-6\left(a^2+b^2\right)x^2+x^4\right]$$

The integrations for the moments are elementary. In each case, the moments may be written in the form

$$\mathbf{E}T^k=\left(\frac{b}{v}\right)^k(m_d(k,\lambda)+m_d(k,-\lambda))$$

where $\lambda=a/b$. The functions $m_d(k,\lambda)$ are listed in the appendix. The first and second moments for all three cases are:

$d=1$:

$$\mathbf{E}T=\frac{b}{v}\left[\frac{1}{4}+\frac{1-\lambda^2}{8\lambda}\log\left(\frac{1+\lambda}{1-\lambda}\right)+\frac{1}{2}\int_0^1\frac{\log x}{\lambda^2x^2-1}dx\right]$$

$$\mathbf{E}T^2=\frac{b^2}{v^2}\left(1-\frac{\lambda^2}{9}\right)$$

$d=2$:

$$\mathbf{E}T=\frac{b}{v}\left[\frac{1}{3}+\frac{\log\left(1-\lambda^2\right)}{3\lambda^2}+\frac{3-\lambda^2}{6\lambda}\log\left(\frac{1+\lambda}{1-\lambda}\right)\right]$$

$$\mathbf{E}T^2=\frac{b^2}{v^2}\left(1-\frac{\lambda^2}{6}\right)$$

$d=3$:

$$\mathbf{E}T=\frac{b}{v}\left[\frac{3\left(1+\lambda^2\right)}{8\lambda^2}-\frac{3\left(1-\lambda^2\right)^2}{16\lambda^3}\log\left(\frac{1+\lambda}{1-\lambda}\right)\right]$$

$$\mathbf{E}T^2=\frac{b^2}{v^2}\left(1-\frac{\lambda^2}{5}\right)$$

Clearly, more complicated position functions $u(t)$ may be substituted in Eq. 3, leading to relatively straight-forward tedious integrations. Note that, depending on the form of $u(t)$, a particle may leave and re-enter H in $(0,t)$.

One modification of practical interest for constant-speed motion is the case of radioactive particles with short lifetimes. Such particles may decay before reaching the boundary of H. Let T_d denote the random lifetime of the particle (time to decay, given birth at point P) and let the random time to reach the boundary in the absence of decay be T_0. If the decay process is independent of the particle's motion, then the particle vanishes unless $t<\min\left(T_0,T_d\right)$. Then

$$\begin{aligned}h(t)&=\mathbf{P}(T_0>t)\mathbf{P}(T_d>t)\\&=\bar{F}_{T_0}(t)\bar{F}_{T_d}(t)\\&=\Omega_{E,H}(vt)\bar{F}_{T_d}(t).\end{aligned}$$

If the particles have fixed lifetime τ, then T_d has degenerate distribution and

$$h(t)=\begin{cases}\Omega_{E,H}(vt) & \text{if } t<\tau\\ 0 & \text{if } t\ge\tau\end{cases}$$

BROWNIAN MOTION

For particles undergoing Brownian motion (with diffusion constant D) the net distance $X(t)$ has probability density function

$$f_X(r,t)=\frac{nC_nr^{n-1}\exp\left(-r^2/4Dt\right)}{\left(4\pi Dt\right)^{n/2}}.$$

One application is the situation where an experimenter can make observations only in a limited field of view (under a microscope, say). An identifiable cell might be observed to be in a certain part of the field of view at time zero and then might be observed again later.

If we expand $F_X(R,t)$ as used in Eq. 2 in a Taylor series in R, we find

$$h(t)=\mathbf{E}_R\left[\sum_{k=0}^{\infty}F_X^{(k)}(0,t)\frac{R^k}{k!}\right]$$
$$=\mathbf{E}_R\left[\sum_{k=0}^{\infty}f_X^{(k-1)}(0,t)\frac{R^k}{k!}\right]$$
$$=\sum_{k=0}^{\infty}f_X^{(k-1)}(0,t)\frac{\mathbf{E}_R R^k}{k!}$$

which gives $h(t)$ in terms of moments of the length of a v-random ray.

Remark. *It seems physically curious that, since $f_X^{(k-1)}(0,t)=0$ for $k<n$, $h(t)$ depends only on moments of R that are of dimensional order or greater.*

If we consider again concentric balls where $E=B_d(a)$ and $H=B_n(b)$, with $d\le n\le 3$ and $a\le b$, then the results of Enns and Ehlers (1993) may be used to evaluate $h(t)$. The expressions are, in general, complicated but not difficult to derive. For the case of $a=b$ and $d=n$ we obtain for $n=2$:

$$h(t)=\frac{1}{\sqrt{\pi}}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}\Gamma\left(k+\frac{1}{2}\right)}{k!(k+1)!}\left(\frac{b^2}{Dt}\right)^k$$

and for $n=3$:

$$h(t)=\frac{3}{\sqrt{\pi}}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(2k+1)(k+1)(k+2)(k-1)!}\left(\frac{b^2}{Dt}\right)^{k+\frac{1}{2}}.$$

Fig. 2 shows how $h(t)$ decreases with time.

For the case of $a<b$ and $d=n=3$ we find

$$h(t)=q(t;a,b)+q(t;-a,b)$$

where

$$q(t;a,b)=\frac{\exp\left(-(b+a)^2/4Dt\right)}{4\sqrt{\pi}a^3}\left[2\left(a^2+b^2-ab\right)\sqrt{4Dt}-\left(4Dt\right)^{3/2}\right]$$
$$+\left(1+\frac{b^3}{a^3}\right)\left[\Phi\left(\sqrt{2}\frac{b+a}{\sqrt{4Dt}}\right)-\frac{1}{2}\right]$$

Here $\Phi(\bullet)$ is the cumulative standard normal distribution function. Fig. 3 shows $h(t)$ for selected values of a with $b=1$.

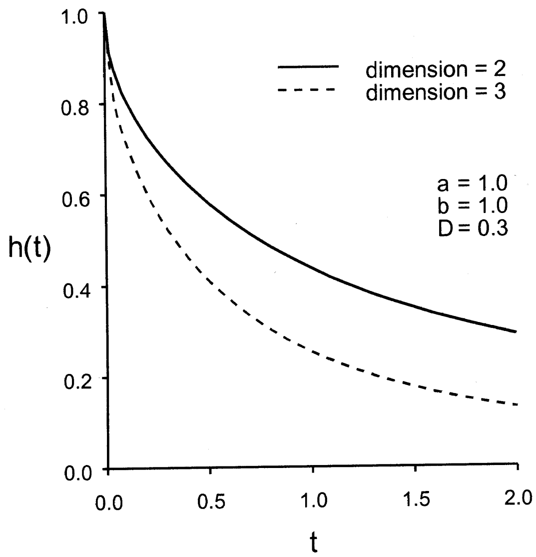

Fig. 2. *The probability h(t) for circular or spherical regions: E = H.*

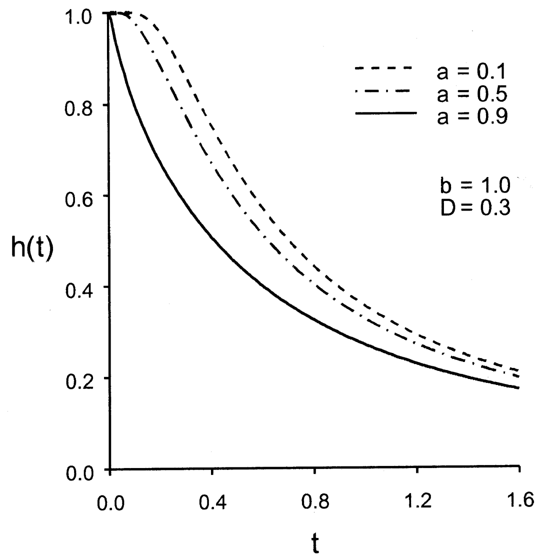

Fig. 3. *The probability h(t) for concentric spherical regions E and H.*

APPENDIX

$$m_1(k,\lambda) = \frac{(1+\lambda)^k(1-k\lambda)}{4k(k^2-1)\lambda} + \frac{1}{4}\sum_{j=0}^{\infty}\frac{\lambda^{2j}}{(2j+1)^2}\left[\binom{k}{2j}+\binom{k-2}{2j}\right]$$

$$m_2(k,\lambda) = \frac{1}{3\lambda^2}\left(\frac{1}{k+2}+\frac{1}{k-1}\right) - \frac{\left(2-3\lambda+\lambda^3\right)\left(1+\lambda\right)^{k-1}}{6\lambda^2(k-1)} + \frac{(1+\lambda)^{k+1}}{2\lambda(k+1)} - \frac{(1+\lambda)^{k+2}}{3\lambda^2(k+2)}$$

$$m_3(k,\lambda) = \frac{3(1+\lambda)^{k+1}}{2(k+3)(k^2-1)}\left(\frac{k+1}{\lambda^2} - \frac{1+\lambda^2}{\lambda^3}\right)$$

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