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THE PIVOTAL TESSELLATION

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ABSTRACT

The tessellation studied here is motivated by some stereological applications of a new expression for the motion invariant density of straight lines in $\mathbb{R}^3$. The term ‘pivotal’ stems from the fact that the tessellation is constructed within a plane which is isotropic through a fixed, ‘pivotal’ origin. Consider either a stationary point process, or a stationary random lattice of points in that plane. Through each point event draw a straight line which is perpendicular to the axis determined by the origin and the point event. The union of all such lines (called p-lines) constitutes the mentioned tessellation. We concentrate on the pivotal tessellation based on a stationary and isotropic planar Poisson point process; we show that this tessellation is not stationary.

Keywords: geometric probability, p-line, pivotal tessellation, Poisson point process, stereology, stochastic geometry.

INTRODUCTION

The purpose of this paper is to explore elementary properties of a special planar tessellation stemming from the application of recent stereological results (Cruz-Orive, 2005; 2008; Gual-Arnau and Cruz-Orive, 2009).

Consider the equatorial disk $B_{2,t} = B_3 \cap L^3_{2[0]}$, where $B_3 \subset \mathbb{R}^3$ represents a ball of radius R centred at the origin O and $L_{2[0]}$ denotes an isotropic plane through O with normal direction $t \in \mathbb{S}^2_+$.

Within the disk $B_{2,t}$, generate N independent and identically distributed uniform random points $\{z_1, z_2, ..., z_N\}$, (Fig. 1a). For each $i = 1, 2, ..., N$, draw a straight line $L_1(z_i)$ through the point z_i and normal to the axis Oz_i. Thus $L_1(z_i)$ is effectively a “point sampled” straight line which will be called a $p-$line. The union of all $p-$lines constitutes a tessellation in the reference disk, (Fig. 1b) which will be called a pivotal tessellation, inasmuch as the containing plane $L_{2[0]}$ can only rotate around a fixed ‘pivot’ O. The practical interest of this construction lies in the following fact. Consider a nonvoid compact subset $Y \subset B_3$ of volume $v_3(Y)$ with piecewise smooth boundary ∂Y of area $v_2(\partial Y)$. Then,

$$\widehat{v}_2(\partial Y) = 2aN^{-1} \sum_{i=1}^{N} v_0 \left\{ (\partial Y \cap B_{2,t}) \cap L_1(z_i) \right\} ,$$
$$\widehat{v}_3(Y) = aN^{-1} \sum_{i=1}^{N} v_1 \left\{ (Y \cap B_{2,t}) \cap L_1(z_i) \right\} , \qquad (1)$$

are unbiased estimators of $v_2(\partial Y)$ and $v_3(Y)$, respectively, where $a := \pi R^2$, and v_0, v_1 denote number of intersections and chord length, respectively (Cruz-Orive, 2005; 2008).

Here we are interested in some properties of the pivotal tessellation constituted by the $p-$lines associated with a planar Poisson point process.

PRELIMINARIES

Given a point $z \in \mathbb{R}^2$ of polar coordinates (ρ, ω), $\rho \in (0, \infty)$, $\omega \in (0, 2\pi)$, we define a p-line $L_1(z)$ as a straight line with normal coordinates (ρ, ω), namely,

$$L_1(z) := L_1(\rho, \omega) = \left\{ x = (x_1, x_2) \in \mathbb{R}^2 : x_1 \cos\omega + x_2 \sin\omega = \rho \right\} . \qquad (2)$$

Consider either a stationary planar point process

$$\Phi = \bigcup_{i \in \mathbb{N}} z_i , \qquad (3)$$

with realizations in $\mathbb{R}^2$, or a stationary random lattice

$$\Lambda_z = \Lambda_0 + z = \bigcup_{i \in \mathbb{N}} z_i , \qquad (4)$$

where $\Lambda_0 \subset \mathbb{R}^2$ is a fixed regular lattice of points and z is a uniform random point in a fundamental tile of Λ_0, (Fig. 3a). In either case, the pivotal tessellation associated with either Φ or Λ_z is

$$\Psi = \bigcup_{i \in \mathbb{N}} L_1(z_i) , \qquad (5)$$

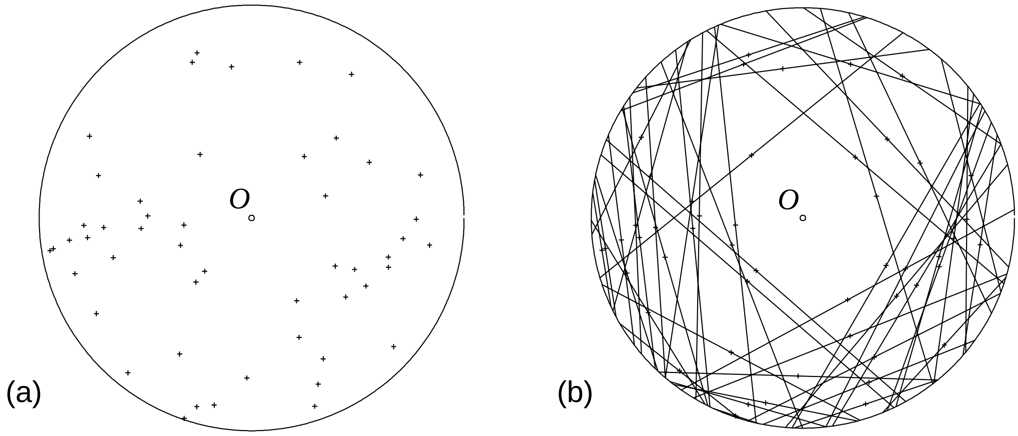

Fig. 1. *(a) Disk centred at O containing 50 independent uniform random point events. (b) Associated pivotal tessellation formed by the corresponding p-lines with respect to O.*

namely the corresponding process of p-lines, (Figs. 1b, 3b).

In this paper $\mathbb{P}(\mathrm{d}x)$ represents the probability element of a random variable X, namely $\mathbb{P}(\mathrm{d}x) := \mathbb{P}(x < X \leq x + \mathrm{d}x)$. If X admits a probability density function $f(x)$, then $\mathbb{P}(\mathrm{d}x) = f(x)\,\mathrm{d}x$. This notation extends to higher dimensions in a natural manner.

THE PIVOTAL POISSON TESSELLATION IN $\mathbb{R}^2$

When the associated process Φ is a stationary and isotropic planar Poisson point process (Stoyan *et al.*, 1995), then the corresponding pivotal tessellation Ψ will be called the pivotal Poisson tessellation. Let τ denote the fixed intensity of Φ, namely,

$$\tau = \frac{\mathbb{E}\{\nu_0(\Phi \cap B)\}}{\nu_2(B)}\,, \quad 0 < \nu_2(B) < \infty\,, \tag{6}$$

where B denotes any subset from the Borel σ-algebra in $\mathbb{R}^2$, and ν_q denotes the q-dimensional Hausdorff measure in $\mathbb{R}^2$, (thus ν_2 represents area, ν_1 curve length, and ν_0 the counting measure).

Next we obtain some properties of the tesselation Ψ. To do this we consider the random intersection

$$\Psi \cap B_R = \bigcup_{i=1}^{N} L_1(z_i) \cap B_R\,, \tag{7}$$

(Fig. 1b), where $B_R \subset \mathbb{R}^2$ is a closed disk of radius R centred at the origin O, whereas $\{z_1, z_2, \ldots, z_N\}$ represent N independent and identically distributed (i.i.d.) uniform random (UR) points in B_R, and

$$N \sim \text{Poisson}(\pi R^2 \tau)\,, \tag{8}$$

so that N is a Poisson random variable with mean and variance equal to $\pi R^2 \tau$.

For a p-line $L_1(z) := L_1(\rho, \omega)$ such that the point z of polar coordinates (ρ, ω) is UR in B_R, it is easy to show that ρ and ω are independent random variables with

$$\begin{aligned}\mathbb{P}(\mathrm{d}\rho) &= 2R^{-2}\rho\,\mathrm{d}\rho\,, \quad 0 < \rho < R\,,\\ \mathbb{P}(\mathrm{d}\omega) &= (2\pi)^{-1}\mathrm{d}\omega\,, \quad 0 < \omega < 2\pi\,.\end{aligned} \tag{9}$$

Lemma 1. *Let $z \in B_R$ denote a UR point in B_R. Then the mean chord length determined in B_R by the corresponding p-line is,*

$$\mathbb{E}\{\nu_1(L_1(z) \cap B_R)\} = \frac{4}{3}R\,. \tag{10}$$

Proof. Straightforward bearing in mind that $\nu_1(L_1(z) \cap B_R) = 2\sqrt{R^2 - \rho^2}$ and using Eq. 9. □

Proposition 1. *The mean total length per unit disk area of the straight line segments determined in B_R by the p-lines of Ψ is*

$$\begin{aligned}\lambda_1^2(R) &:= (\pi R^2)^{-1}\mathbb{E}\{\nu_1(\Psi \cap B_R)\}\\ &= \frac{4}{3}\tau R\,.\end{aligned} \tag{11}$$

Proof. Conditional on the number N of p-lines from Ψ hitting B_R, by Eq. 10 we have,

$$\mathbb{E}\{\nu_1(\Psi \cap B_R)|N\} = \frac{4}{3}RN\,. \tag{12}$$

Using the premise (Eq. 8) and dividing by πR^2, the result follows. □

Consequence. *From Eq. 11 we see that* $\lambda_1^2(R) = O(R)$*, which implies that the pivotal Poisson tessellation* Ψ *associated with* Φ *is not stationary.*

Remark 1. *The mean area of the equatorial disk* $B_{2,t}$ *considered in the Introduction (see Fig. 2) per unit volume of the corresponding ball* B_3*, is* $\lambda_2^3(R) = \pi R^2/(4\pi R^3/3) = 3/(4R)$*. On the other hand, the mean total chord length of the bounded pivotal Poisson tessellation in* $B_{2,t}$*, per unit area of* $B_{2,t}$*, is given by Eq. 11. Therefore, the mean total chord length of such planar tessellation per unit volume of the reference ball* B_3*, is*

$$\lambda_1^3(R) = \lambda_1^2(R) \cdot \lambda_2^3(R) = \tau \,, \tag{13}$$

namely a constant. This result is consistent with the fact that p-lines are effectively motion invariant in $\mathbb{R}^3$*.*

Lemma 2. *Let* z_1, z_2 *denote two i.i.d. UR points in* B_R*. Then,*

$$\mathbb{P}\{L_1(z_1) \cap L_2(z_2) \in B_R\} = \frac{3}{8}, \quad \forall R \in (0,\infty) \,, \tag{14}$$

that is, the probability that the corresponding two p-lines intersect inside B_R *is a known constant equal to 3/8 for any* $R > 0$*.*

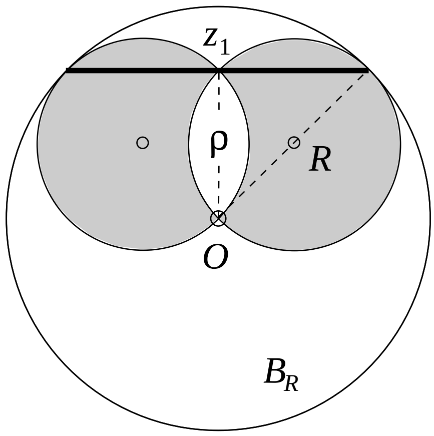

Fig. 2. *The probability that a p-line* $L_1(z_2)$ *associated with a UR point* $z_2 \in B_R$ *hits a given p-chord* $L_1(z_1) \cap B_R$ *(thick straight line segment in the figure) is equal to the probability that* z_2 *falls in the support set (shaded region) of the given p-chord with respect to O.*

Proof. Fix one of the two points, *e.g.*, $z_1 = (\rho,\omega)$, $\rho \in (0,R)$, $\omega \in (0,2\pi)$, and denote by $p(\rho,\omega;R)$ the required probability conditional on (ρ,ω). By the definition of support set (Cruz-Orive, 2005; Gual-Arnau and Cruz-Orive, 2009), it follows that $p(\rho,\omega;R)$ is the probability that z_2 falls in the support set $H_{L_1(z_1)\cap B_R}$ of the chord $L_1(z_1) \cap B_R$ with respect to the disk centre O (see Fig. 2). Bearing in mind that $\mathbb{P}(\mathrm{d}z_2) = (\pi R^2)^{-1}\mathrm{d}z_2$, $z_2 \in B_R$, we have

$$\begin{aligned} p(\rho,\omega;R) &:= \mathbb{P}\{L_1(z_1) \cap L_2(z_2) \in B_R | \rho,\omega\} \\ &= \int_{H_{L_1(z_1)\cap B_R}} \mathbb{P}(\mathrm{d}z_2) \\ &= \frac{1}{2} - \frac{2}{\pi} \cdot g_{\mathrm{disk}}\left(\sqrt{1-\rho^2/R^2}\right), \end{aligned} \tag{15}$$

where

$$g_{\mathrm{disk}}(x) = \frac{1}{2}\left(\cos^{-1} x - x\sqrt{1-x^2}\right) \,, \quad (0 \le x \le 1) \,, \tag{16}$$

is the geometric covariogram of a disk of unit diameter. It is readily verified that,

$$\begin{aligned} \mathbb{P}\{L_1(z_1) \cap L_2(z_2) \in B_R\} &= \int_0^R p(\rho,\omega;R)\,\mathbb{P}(\mathrm{d}\rho) \\ &= \frac{3}{8} \,, \end{aligned} \tag{17}$$

where $\mathbb{P}(\mathrm{d}\rho)$ is given by the first Eq. 9. □

Proposition 2. *Let* $\lambda_0^{(0)}(R)$*,* $\lambda_0^{(1)}(R)$*, and* $\lambda_0^{(2)}(R)$ *denote the mean total numbers per unit disk area of the vertices, edges and connected regions constituting the bounded tessellation* $\Psi \cap B_R$*, respectively. Then,*

$$\begin{aligned} \lambda_0^{(0)}(R) &= \frac{3\pi}{16}\tau^2 R^2 + 2\tau \,, \\ \lambda_0^{(1)}(R) &= \frac{3\pi}{8}\tau^2 R^2 + 3\tau \,, \\ \lambda_0^{(2)}(R) &= \frac{3\pi}{16}\tau^2 R^2 + \tau + \frac{1}{\pi R^2} \,, \end{aligned} \tag{18}$$

where the terms following the first one in the right hand side of the preceding identities represent the contributions of the disk boundary ∂B_R*.*

Proof. We use the method of Santaló (1940; 1976 p. 51). Conditional on the number N of p-lines from Ψ hitting B_R, let $V_{B^o}(N), V_{\partial B}(N)$ denote the mean number of vertices interior to B_R and in ∂B_R, respectively, and set $V_{B^o}(N) + V_{\partial B}(N) = V(N)$. Then using Lemma 2,

$$\mathbb{E}\{V(N)|N\} = \binom{N}{2}\frac{3}{8} + 2N \,. \tag{19}$$

Likewise, let $E_{B^o}(N)$, $E_{\partial B}(N)$ denote the mean number of edges interior to B_R and in ∂B_R, respectively, and set $E_{B^o}(N) + E_{\partial B}(N) = E(N)$. At each interior vertex there meet 4 edges, but they are counted twice because each edge has two vertices as endpoints. On the other hand, at each boundary vertex

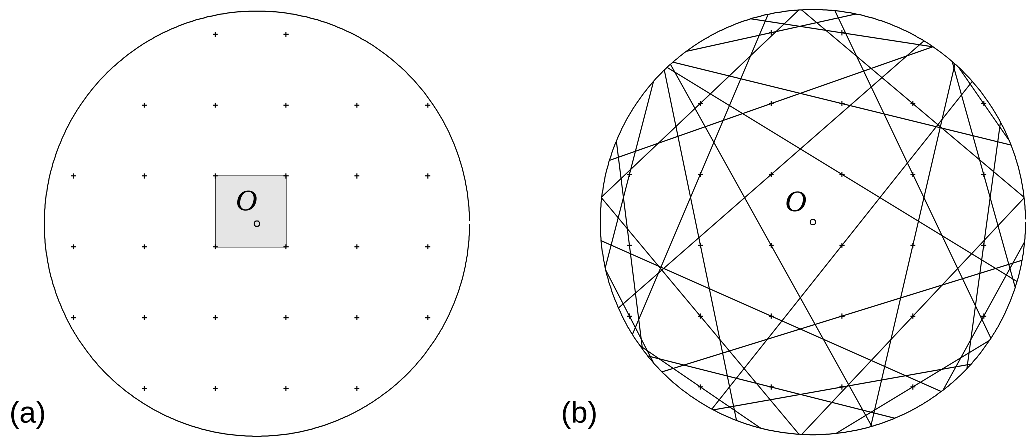

Fig. 3. *(a) A realization of a stationary random square lattice inside a disk. The centre O of the disk is uniform random in a tile of the lattice (shaded square). (b) The associated pivotal lattice tessellation inside the disk, which is in fact the kind of probe used in the applications (Cruz-Orive, 2005; 2008).*

there meet 3 edges, but they are also counted twice for the same reason. Therefore,

$$\mathbb{E}\{E(N)|N\} = 2\binom{N}{2}\frac{3}{8} + 3N\,. \tag{20}$$

Finally, let $F(N)$ denote the total number of connected regions or "faces". By Euler's formula we have $V(N)+F(N)-E(N)=1$, and therefore,

$$\mathbb{E}\{F(N)|N\} = \binom{N}{2}\frac{3}{8} + N + 1\,. \tag{21}$$

Taking expectations on both sides of each of the identities (Eqs. 19–21) with respect to N, bearing Eq. 8 in mind, and dividing by πR^2 in each case, the corresponding identities (Eq. 18) are obtained. □

Definition. *The mean number of vertices (or of sides), the mean boundary length, and the mean area of a connected region from the bounded tessellation $\Psi \cap B_R$, are defined respectively as follows,*

$$\begin{aligned} \mathbb{E}\{N(R)\} &= \frac{2\lambda_0^{(1)}(R)}{\lambda_0^{(2)}(R)}\,, \\ \mathbb{E}\{B(R)\} &= \frac{2\lambda_1^2(R) + 2/R}{\lambda_0^{(2)}(R)}\,, \\ \mathbb{E}\{A(R)\} &= \frac{1}{\lambda_0^{(2)}(R)}\,. \end{aligned} \tag{22}$$

Proposition 3. *The characteristics given in the preceding definition satisfy the following asymptotic relations,*

$$\begin{aligned} \mathbb{E}\{N(R)\} &= 4 + O(R^{-2})\,, \\ \mathbb{E}\{B(R)\} &= O(R^{-1})\,, \\ \mathbb{E}\{A(R)\} &= O(R^{-2})\,. \end{aligned} \tag{23}$$

Proof. Substitute the results Eq. 11 and Eq. 18 into Eq. 22. □

CONCLUSIONS AND COMMENTS

Concerning the planar pivotal Poisson tessellation Ψ, the main conclusion is that it is not stationary, as illustrated by the results (Eqs.11, 18, and 23). The asymptotic mean number of vertices of a polygon is 4, as in the ordinary Poisson tessellation of straight lines (Stoyan *et al.*, 1995), but the remaining properties change with the distance from the origin. The non stationarity is intuitively plausible on seeing Fig. 1b. A priori one might think that, because p-lines on an isotropic plane $L_{2[0]}$ are effectively motion invariant in $\mathbb{R}^3$, and because the associated point process is stationary Poisson, then Ψ would also be stationary in $\mathbb{R}^2$, but this is not the case. As confirmed by Eq. 13, the length density of the $p-$lines of the planar pivotal Poisson tessellation must be constant in $\mathbb{R}^3$ because they are motion invariant in $\mathbb{R}^3$. Note that the plane $L_{2[0]}$ is less and less "dense" away from the origin; this effect must be compensated by a higher and higher line length density in that plane away from the origin, and this is indeed what happens.

For estimation purposes via Eq. 1 it is simpler and more efficient to start with a stationary random lattice of points (Fig. 3a), instead of a Poisson point process. The corresponding pivotal lattice tessellation (Fig. 3b) will enjoy similar properties. An exact study of the latter might be prohibitive, however, because the number of lattice points inside a disk is a complicated oscillating function of the disk diameter.

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