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CONVEX BODIES AND GAUSSIAN PROCESSES

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ABSTRACT

For several decades, the topics of the title have had a fruitful interaction. This survey will describe some of these connections, including the GB/GC classification of convex bodies, Ito-Nisio singularities from a geometric viewpoint, Gaussian representation of intrinsic volumes, the Wills functional in a Gaussian context, and inequalities.

Keywords: convex body, Gaussian process, intrinsic volume, random convex body, strong law of large numbers for random convex bodies, Wills functional.

INTRODUCTION

For several decades, the topics of the title have had a fruitful interaction. This survey will describe some of these connections, including the GB/GC classification of convex bodies, Ito-Nisio singularities from a geometric viewpoint, Gaussian representation of intrinsic volumes, the Wills functional in a Gaussian context, and inequalities. For fuller discussions and references, the interested reader is urged to consult the bibliography.

GEOMETRIC PRELIMINARIES AND NOTATION

The setting is either finite dimensions or infinite dimensions, that is, $\boldsymbol{R}^d$ or ℓ_2. Schneider (1993) gives an excellent treatment of the classical theory of convex bodies. The following items and notation will be appear:

- Convex bodies $\mathscr{K}$: compact, convex $K, L, \ldots$
- Scaling: $\lambda K = \{\lambda x : x \in K\}$.
- Minkowski addition: $K + L = \{x + y : x \in K, y \in L\}$.
- Closed unit ball: B, B_d .
- λ-parallel body: $K + \lambda B$.
- Support function: $h_K(x) = \sup_{t \in K} <x, t>$.
- Hausdorff metric:

$$\begin{aligned}\rho(K,L) &= \inf\{\lambda > 0 : K \subseteq L + \lambda B, L \subseteq K + \lambda B\} \\ &= \sup_{\|u\|=1} |h_K(u) - h_L(u)| .\end{aligned}$$

- Norm: $\|K\| = \max_{x \in K} \|x\| = \max_{\|u\|=1} h_K(u)$.

GAUSSIAN PROCESSES WITH ISONORMAL INDEXING

For background and references on Gaussian processes, one can consult, for example, Lifshits (1995) and Bogachev (1998). We assume throughout a sequence of independent standard (*i.e.*, $N(0,1)$) Gaussian random variables:

$$Z = (Z_1, Z_2, \ldots) .$$

For a convex body $K \subset \ell_2$ and $t \in K$, we consider the map

$$t \mapsto X_t = <t, Z> = \sum_{i=1}^{\infty} t_i Z_i .$$

The image is an $N(0, \|t\|^2)$ variable, and the collection $\{X_t, t \in K\}$ is called an *isonormally-indexed* Gaussian process in view of the isometric-isomorphism:

$$K \longleftrightarrow \{X_t,\ t \in K\} .$$

Specifically,

$$\begin{gathered} at + b\hat{t} \leftrightarrow aX_t + bX_{\hat{t}} \\ \|t - \hat{t}\|^2 = E\,(X_t - X_{\hat{t}})^2 \ . \end{gathered}$$

Another key point is the identification

$$h_K(Z) = \sup_{t \in K} <t, Z> = \sup_{t \in K} X_t \ .$$

LIMIT THEOREMS

Consider a *random* convex body X, which is a measurable map from a probability space to its space of values endowed with the Hausdorff metric:

$$X : \{\Omega, \mathscr{F}, P\} \longrightarrow (\mathscr{K}, \rho).$$

If X is bounded in expected norm, $E\|X\| < \infty$, then one has an expectation $EX \in \mathscr{K}$, which can be given implicitly in terms of its support function

$$h_{EX}(\cdot) = Eh_X(\cdot).$$

There is a strong law of large numbers:

Theorem 1 (Artstein and Vitale, 1975) *If $X_1, X_2, \ldots$ are independent and identically distributed random convex bodies with $E\|X_1\| < \infty$, then*

$$\overline{X_n} = \frac{X_1 + X_2 + \cdots + X_n}{n} \xrightarrow{\text{a.s.}} EX_1.$$

The formulation of an accompanying central limit theorem takes into account that there is no convenient notion of subtraction for convex bodies, and so the identification with support functions is used:

Theorem 2 (Weil, 1982) *If $X_1, X_2, \ldots$ are iid and $E\|X\|^2 < \infty$, then $\sqrt{n}\left[h_{\overline{X_n}}(u) - h_{EX_1}(u)\right]$ converges to a centered Gaussian process with inherited covariance function.*

A different kind of limit theorem appears in Bonetti and Vitale (2000).

THE STEINER FORMULA AND INTRINSIC VOLUMES

The Steiner formula for the volume of the parallel body to a convex body in $\boldsymbol{R}^d$ is

$$\mathrm{vol}_d(K + \lambda B) = \sum_{j=0}^{d} \mathrm{vol}_j(B_j)\lambda^j\, V_{d-j}(K),$$

where the constants $V_j(K)$, $j = 0, 1, \ldots, d$ are known as *intrinsic volumes*.

Following Vitale (1995), we give a derivation of the formula, which also serves to display the nature of the intrinsic volumes: consider *iid* isotropic line segments $L_1, \ldots, L_n$, such that $EL_1 = B_d$. By the strong law of large numbers,

$$(1/n)(L_1 + \cdots + L_n) \to B_d$$

as $n \to \infty$, and so

$$\mathrm{vol}_d\left[K + (\lambda/n)(L_1 + \cdots + L_n)\right] \to \mathrm{vol}_d(K + \lambda B_d).$$

For one line segment (*i.e.*, $n = 1$), one has

$$\mathrm{vol}_d(K + \lambda L_1) = \mathrm{vol}_d(K) + \lambda|L_1| \cdot \mathrm{vol}_{d-1}(\Pi_{L_1^\perp} K),$$

where $\Pi_{L_1^\perp}$ signifies projection onto the subspace orthogonal to the one spanned by L_1. By induction,

$$\mathrm{vol}_d\left[K + (\lambda/n)(L_1 + \cdots + L_n)\right] = \sum_{\substack{S \subseteq \{1,2,\ldots,n\} \\ 0 \le |S| \le d}} (\lambda/n)^{|S|} \mathrm{vol}_{|S|}(L_S)\, \mathrm{vol}_{d-|S|}\left(\Pi_{L_S^\perp} K\right),$$

where $L_S = \sum_{i \in S} L_i$. This can be re-expressed as

$$\mathrm{vol}_d\left[K + (\lambda/n)(L_1 + \cdots + L_n)\right] = \sum_{j=0}^{d} \frac{\binom{n}{j}}{n^j} \lambda^j U_{jn},$$

where

$$U_{jn} = \frac{1}{\binom{n}{j}} \sum_{|S|=j} \mathrm{vol}_j(L_S)\, \mathrm{vol}_{d-j}\left(\Pi_{L_S^\perp} K\right)$$

has the form of a *U-statistic*. Then

$$\begin{aligned}
\mathrm{vol}_d(K + \lambda B_d) &= \lim_n \mathrm{vol}_d\left[K + (\lambda/n)(L_1 + \cdots + L_n)\right] \\
&= \lim_n \sum_{j=0}^{d} \frac{\binom{n}{j}}{n^j} \lambda^j U_{jn} \\
&= \sum_{j=0}^{d} \frac{\lambda^j}{j!} \lim_n U_{jn} \\
&= \sum_{j=0}^{d} \frac{\lambda^j}{j!} c_{d-j} E\mathrm{vol}_{d-j}\left(\Pi_{d-j} K\right) \\
&= \sum_{j=0}^{d} \mathrm{vol}_j(B_j)\lambda^j\, V_{d-j}(K),
\end{aligned}$$

where Π_j signifies projection onto a random subspace of dimension j and

$$V_j(K) = \binom{d}{j} \frac{\mathrm{vol}_d(B_d)}{\mathrm{vol}_j(B_j)\mathrm{vol}_{d-j}(B_{d-j})} E\mathrm{vol}_j(\Pi_j K). \tag{1}$$

A Gaussian version, shown below in Eq. 2, follows from noticing that, in Eq. 1, a key property is that, for an independent, random orthogonal O,

$$\Pi_j O \stackrel{\mathrm{d}}{=} \Pi_j.$$

It is also true that

$$Z_{[j,d]}O \stackrel{d}{=} Z_{[j,d]},$$

where $Z_{[j,d]}$,is a $j \times d$ matrix of independent $N(0,1)$ variables. This can be used (Vitale, 2008) to show

$$V_j(K) = \frac{(2\pi)^{j/2} E\mathrm{vol}_j\left(Z_{[j,d]}K\right)}{j!\,\mathrm{vol}_j(B_j)} \quad . \tag{2}$$

Next we identify some of the intrinsic volumes:

$$\begin{aligned}
V_0(K) &= 1\\
V_1(K) &= \text{intrinsic width}\\
&= \sqrt{2\pi} E h_K(Z) = \sqrt{2\pi}\, E \sup_{t\in K} X_t\\
&\vdots\\
V_{d-1}(K) &= 1/2 \cdot \text{surface area of } K\\
V_d(K) &= d\text{-dimensional volume of } K\\
V_j(K) &= 0 \quad \text{for} \quad j > d\\
V_j\left(\prod_1^n [a_i,b_i]\right) &= \sum_{i_1<i_2<\cdots<i_j} (b_{i_1}-a_{i_1})\cdots(b_{i_j}-a_{i_j})\\
V_1(B_d) &\sim \sqrt{2\pi d}.
\end{aligned}$$

EXTENSION OF INTRINSIC VOLUMES TO CONVEX BODIES IN ℓ_2

The extension of intrinsic volumes to convex bodies in Hilbert space and specifically to ℓ_2 was undertaken by Sudakov (1971) and Chevet (1976). To begin, let us identify the following collections:

$$\begin{aligned}
\mathscr{K}_{\mathrm{d}} &= \text{convex bodies in } \boldsymbol{R}^d\\
\mathscr{K} &= \text{convex bodies in } \ell_2\\
\mathscr{K}_{\mathrm{FD}} &= \text{finite–dimensional convex bodies in } \ell_2.
\end{aligned}$$

In view of the monotonicity of the intrinsic volumes under set inclusion, it is natural to extend them to infinite dimensional convex bodies as follows: for arbitrary $K \in \mathscr{K}$, define

$$\begin{aligned}
V_j(K) &= \sup\{V_j(\widehat{K}) : \widehat{K} \subseteq K,\ \widehat{K} \in \mathscr{K}_{\mathrm{FD}}\}\\
\mathscr{K}_{\mathrm{GB}} &= \{K \in \mathscr{K} : V_1(K) < \infty\}.
\end{aligned}$$

GB stands for “Gaussian Bounded” (Dudley, 1967) and refers to the following identification.

Theorem 3 $K \in \mathscr{K}_{\mathrm{GB}} \iff \{X_t, t \in K\}$ *is an almost–surely bounded Gaussian process. That is* $P(\sup_{t\in K_0} X_t < \infty) = 1$ *for any denumerable subset* $K_0 \subset K$.

The following also hold:

1. $\mathscr{K}_{\mathrm{FD}} \subset \mathscr{K}_{\mathrm{GB}} \subset \mathscr{K}$.
2. $K \in \mathscr{K}_{\mathrm{GB}} \Rightarrow V_j(K) < \infty, \quad j = 2,3,\ldots$
3. $K \in \mathscr{K}_{\mathrm{GB}} \Rightarrow V_j(K) = \dfrac{(2\pi)^{j/2} E\mathrm{vol}_j\left(Z_{[j,\infty]}K\right)}{j!\,\mathrm{vol}_j(B_j)}$, where $Z_{[j,\infty]}$ is a $j \times \infty$ matrix of independent $N(0,1)$ variables. Equivalently (Tsirel’son, 1985),

$$V_j(K) = \frac{(2\pi)^{j/2} E\mathrm{vol}_j\left(\{(X_t^1, X_t^2, \ldots, X_t^j), t \in K\}\right)}{j!\,\mathrm{vol}_j(B_j)}.$$

Some canonical cases are given in the next example.

Example Given a decreasing sequence of positive constants $\{a_n\}$ and an orthonormal set $\{e_n,\ n = 1,2,\ldots\}$, set

$$K = \overline{\mathrm{conv}}\{a_n e_n,\ n = 1,2,\ldots\}.$$

Then

$$\begin{aligned}
K \in \mathscr{K} &\iff a_n \downarrow 0,\\
K \in \mathscr{K}_{\mathrm{FD}} &\iff a_n = 0 \ \text{eventually},\\
K \in \mathscr{K}_{\mathrm{GB}} &\iff a_n = \mathscr{O}\left[(\log n)^{-1/2}\right].
\end{aligned}$$

Example An example of an infinite-dimensional convex body that has no finite-dimensional analogue is as follows. Consider a map

$$f : [0,1] \to \mathscr{H} \text{ (Hilbert space)}$$

that satisfies

1. $0 \le x_1 \le x_2 \le x_3 \le x_4 \le 1 \Rightarrow$
$$[f(x_2) - f(x_1)] \perp [f(x_4) - f(x_3)]$$
2. $\|f(x_2) - f(x_1)\|^2 = |x_2 - x_1|$ for all $0 \le x_1 \le x_2 \le 1$.

The associated *Brownian Motion Body* is defined to be

$$\overline{\mathrm{conv}}\{f([0,1])\} \subset \mathscr{H}.$$

All Brownian motion bodies are the same, of course, up to an isometry. A particular realization in $L^2[0,1]$ is

$$\{g : [0,1] \to \boldsymbol{R}^1 \mid 0 \le g \le 1,\ g \uparrow\}.$$

Theorem 4 (Gao and Vitale, 2001)

$$V_j(BMB) = \frac{\mathrm{vol}_j(B_j)}{j!} \qquad j = 1,2,\ldots$$

SINGULARITIES

Although intrinsic volumes are defined, and finite, for all GB convex bodies, they are not continuous. That is, it is possible to have GB bodies with $K_n \to K$, but $V_j(K_n) \not\to V_j(K)$. In particular, one can have

$$K_n \downarrow \{p\}, \text{ but } V_1(K_n) \not\downarrow 0 = V_1(\{p\}).$$

This leads to the following definition.

Definition $t^* \in K \in \mathscr{K}_{\text{GB}}$ is a *singularity* of K if

$$V_1(K \cap B(t^*, \varepsilon)) \not\downarrow 0 \qquad \text{as } \varepsilon \to 0.$$

Definition $\mathscr{K}_{\text{GC}} = \{K \in \mathscr{K}_{\text{GB}} : K$ has no singularities$\}$.

One has

$$\mathscr{K}_{\text{FD}} \subset \mathscr{K}_{\text{GC}} \subset \mathscr{K}_{\text{GB}} \subset \mathscr{K}.$$

GC stands for “Gaussian Continuous” (Dudley, 1967), and the following gives the connection.

Theorem 5 $K \in \mathscr{K}_{\text{GC}} \iff \{X_t, t \in K\}$ *is an almost–surely continuous Gaussian process. That is* $t_n \to t \Rightarrow X_{t_n} \to X_t$ *almost–surely.*

Example (continued)

$$K = \overline{\text{conv}}\{a_n e_n,\ n = 1, 2, \ldots\} \in \mathscr{K}_{\text{GC}}$$

$$\iff a_n = o\left[(\log n)^{-1/2}\right].$$

ITO-NISIO THEORY

Theorem 6 (Ito and Nisio, 1969) *Suppose that* $t^* \in K \in \mathscr{K}_{\text{GB}}$. The oscillation of X at t^*, *given by*

$$0 \le 2 \cdot \text{osc}(t^*) = \lim_{\varepsilon \downarrow 0} \left[\sup_{t \in K \cap B(t^*, \varepsilon)} X_t - \inf_{t \in K \cap B(t^*, \varepsilon)} X_t \right],$$

is almost surely constant. Further, $\text{osc}(t^*) > 0 \iff t^*$ *is a singularity of* K.

The following elaborates this observation.

Theorem 7 (Vitale, 2001) *Suppose that* $\text{osc}\,(t^*) > 0$. *Then*

1. $\text{osc}(t^*) \overset{\text{a.s.}}{=} \frac{1}{\sqrt{2\pi}} \lim_{\varepsilon \downarrow 0} V_1(K \cap B(t^*, \varepsilon))$.
2. *For each* j,
$$\lim_{\varepsilon \downarrow 0} V_j(K \cap B(t^*, \varepsilon)) > 0. \tag{3}$$
3. $K \cap B(t^*, 0+) \approx \frac{1}{\sqrt{2 \cdot \pi \cdot \infty}} B_\infty(t^*, \text{osc}\,(t^*))$ *in the sense that for each* j, *the limit in Eq. 3 is equal to*
$$\lim_{d \to \infty} V_j \left(\frac{1}{\sqrt{2 \cdot \pi \cdot d}}\, B(t^*, \text{osc}\,(t^*)) \right)$$
(both being $\dfrac{\text{osc}^{\,j}(t^*)}{j!}$*).*
4. *Define* $\text{osc}\,(K) = \sup\{E\,\text{osc}\,(t^*) : t^* \in K\}$. *Then*
$$\text{osc}\,(K) = \lim_{j \to \infty} \frac{(j+1)V_{j+1}(K)}{V_j(K)}.$$

THE WILLS FUNCTIONAL AND BOUNDS FOR GAUSSIAN PROCESSES

In the context of a question in lattice point enumeration, Wills (1973) defined the following functional. It has come to play an important role in the connection between the theories of convex bodies and Gaussian processes.

$$W(K) = \int_{R^d} \frac{1}{(2\pi)^{d/2}} e^{-\frac{1}{2}\text{dist}^2(x,K)}\, dx.$$

An alternate representation can be derived as follows:

$$\begin{aligned}
W(K) &= \frac{1}{(2\pi)^{d/2}} \int_{R^d} P(\text{dist}(x,K) \le \Lambda)\, dx, \\
&\qquad\qquad \text{where } f_\Lambda(\lambda) = 1(\lambda \ge 0)\lambda e^{-(1/2)\lambda^2} \\
&= \frac{1}{(2\pi)^{d/2}} \int_{R^d} E1(\text{dist}(x,K) \le \Lambda)\, dx \\
&= \frac{1}{(2\pi)^{d/2}}\, E \int_{R^d} 1(\text{dist}(x,K) \le \Lambda)\, dx \\
&= \frac{1}{(2\pi)^{d/2}}\, E\, \text{vol}(K + \Lambda B). \\
&= \frac{1}{(2\pi)^{d/2}}\, E \left[\sum_{i=0}^{d} \text{vol}_i(B_i) \Lambda^i V_{d-i}(K) \right] \\
&= \frac{1}{(2\pi)^{d/2}} \sum_{i=0}^{d} \text{vol}_i(B_i) \left[E\Lambda^i \right] V_{d-i}(K) \\
&= \frac{1}{(2\pi)^{d/2}} \sum_{i=0}^{d} \text{vol}_i(B_i) \left[\frac{(2\pi)^{i/2}}{\text{vol}_i(B_i)} \right] V_{d-i}(K) \\
&= \sum_{j=0}^{d} \frac{1}{(2\pi)^{j/2}} V_j(K).
\end{aligned}$$

A second alternate representation proceeds as follows:

$$\begin{aligned} W(K) &= \int_{R^d} \frac{1}{(2\pi)^{d/2}} e^{-\frac{1}{2}\mathrm{dist}^2(x,K)} dx \\ &= \int_{R^d} \frac{1}{(2\pi)^{d/2}} e^{-\frac{1}{2}\inf_{t\in K}\|x-t\|^2} dx \\ &= \int_{R^d} e^{\sup_{t\in K}[<x,t>-\frac{1}{2}\|t\|^2]} \frac{1}{(2\pi)^{d/2}} e^{-\frac{1}{2}\|x\|^2} dx \\ &= E e^{\sup_{t\in K}[X_t-\frac{1}{2}\sigma_t^2]} . \end{aligned}$$

One then has

$$E e^{\sup_{t\in K}[X_t-\frac{1}{2}\sigma_t^2]} = \sum_{j=0}^{d} \frac{1}{(2\pi)^{j/2}} V_j(K) .$$

This can be extended by writing rK, $r > 0$, in place of K, which itself can be taken to be an element of $\mathscr{K}_{\mathrm{GB}}$:

$$E e^{\sup_{t\in K}[rX_t-\frac{1}{2}r^2\sigma_t^2]} = \sum_{j=0}^{\infty} \left(\frac{r}{\sqrt{2\pi}}\right)^j V_j(K) .$$

The Alexandrov-Fenchel inequality (Schneider, 1993) implies that

$$V_j(K) \le \frac{1}{j!} V_1^j(K) = \frac{1}{j!} \left[\sqrt{2\pi} E \sup_{t\in K} X_t\right]^j ,$$

and hence

$$\sum_{j=0}^{\infty} \left(\frac{r}{\sqrt{2\pi}}\right)^j V_j(K) \le e^{r E \sup_{t\in K} X_t} .$$

Thus we have shown the following:

Theorem 8 (Tsirel'son, 1985; Vitale, 1996, 2001) *If* $\{X_t,\, t\in K\}$ *is a mean-zero, bounded Gaussian process, then*

$$E e^{\sup_t\{X_t-(1/2)EX_t^2\}} \le e^{E\sup_t X_t} .$$

An immediate consequence is a deviation bound:

Theorem 9 (Pisier, 1986; Vitale, 1996, 1999)

$$P(\sup_t X_t - E\sup_t X_t \ge a) \le \exp[-(1/2)(a^2/\sigma^2)]. \quad (4)$$

Proof: Set $\sigma^2 = \sup_{t\in K}\sigma_t^2$. It is direct to show

$$E e^{r[\sup_{t\in K} X_t - E\sup_{t\in K} X_t]} \le e^{\frac{1}{2}r^2\sigma^2} .$$

Then

$$\begin{aligned} &P(\sup_{t\in K} X_t - E\sup_{t\in K} X_t \ge a) = \\ &\qquad = P(r[\sup_{t\in K} X_t - E\sup_{t\in K} X_t] \ge ra) \\ &\qquad = P(e^{r[\sup_{t\in K} X_t - E\sup_{t\in K} X_t]} \ge e^{ra}) \\ &\qquad \le E e^{r[\sup_{t\in K} X_t - E\sup_{t\in K} X_t]} e^{-ra} \\ &\qquad \le e^{\frac{1}{2}r^2\sigma^2-ra} , \end{aligned}$$

which is minimized at $r = a/\sigma^2$, and the assertion follows.

CONCLUSION

We have surveyed topics that combine elements of the theory of convex bodies and Gaussian processes.

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