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SUPPLEMENTARY MATERIALS – FUNCTIONAL ASPLUND METRICS FOR PATTERN MATCHING, ROBUST TO VARIABLE LIGHTING CONDITIONS

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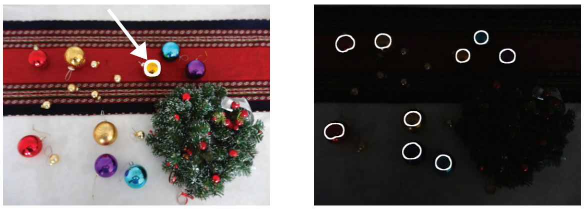
This document corresponds to the supplementary materials of the article. It is organised as follows: i) a video abstract, ii) the proofs of some propositions, iii) the verification of the metric properties, iv) the invariances of the robust to noise metrics and v) details about the illustration section.

VIDEO ABSTRACT

A video is available as *graphical abstract*.



A template is selected in a high-contrast image.

Similar objects to the template are detected in a low-contrast image with an Asplund metric.

Fig. 1. *Cover graphic of the video abstract.*

PROOF OF PROPOSITIONS 2, 3 AND 4

Noyel and Jourlin (2017) have introduced the following proofs which correspond to the map of LIP[1]-multiplicative Asplund distances.

Proof of proposition 2.

Using Eq. 14, $\forall x \in D$, $\forall h \in D_b$, $\forall \alpha \in \mathbb{R}^+$, there is:

$$\begin{aligned}
&\alpha(x) \triangle\!\!\!\times b(h) \geq f(x+h)\\
&\Leftrightarrow M(1-(1-b(h)/M)^{\alpha(x)}) \geq f(x+h), \text{ (Eq. 3)}\\
&\Leftrightarrow \alpha(x) \geq \frac{\ln(1-f(x+h)/M)}{\ln(1-b(h)/M)},
\end{aligned}$$

where $(1-\frac{b(h)}{M}) \in]0,1[$ and $\ln\left(1-\frac{b(h)}{M}\right) < 0$.

[1]Abbreviation: LIP (Logarithmic Image Processing).

With $\tilde{f} = \ln(1-f/M)$, Eq. 14 becomes:

$$\begin{aligned}
\lambda_b f(x) &= \inf\{\alpha(x), \alpha(x) \geq (\tilde{f}(x+h)/\tilde{b}(h)), h \in D_b\}\\
&= \bigvee\{\tilde{f}(x+h)/\tilde{b}(h), h \in D_b\}.
\end{aligned}$$

The last equality is due to the complete lattice structure. In a similar way, Eq. 15 becomes:

$$\begin{aligned}
\mu_b f(x) &= \sup\{\alpha(x), \alpha(x) \leq (\tilde{f}(x+h)/\tilde{b}(h)), h \in D_b\}\\
&= \bigwedge\{\tilde{f}(x+h)/\tilde{b}(h), h \in D_b\}.
\end{aligned}$$

□

Proof of proposition 3.

Let $b = b_0 \in (\mathcal{T}^*)^{D_b}$ a flat structuring element ($\forall x \in D_b$, $b(x) = b_0$). Eq. 16 of λ_b and Eq. 17 of μ_b can be simplified as follows:

$$\begin{aligned}
\lambda_{b_0} f(x) &= (1/\tilde{b}_0)\bigwedge\{\tilde{f}(x+h), h \in D_b\}, \text{ because } \tilde{b}_0 < 0\\
&= (1/\tilde{b}_0)\ln[1-(\bigvee\{f(x-h), -h \in D_b\})/M]\\
&= (1/\tilde{b}_0)\ln[1-(\delta_{\overline{D}_b} f(x))/M].
\end{aligned}$$

The infimum $\bigwedge$ is changed into a supremum $\bigvee$ because the function $\tilde{f} : x \to \ln(1-x/M)$ is a continuous decreasing mapping. Similarly,

$$\begin{aligned}
\mu_{b_0} f(x) &= (1/\tilde{b}_0)\bigvee\{\tilde{f}(x+h), h \in D_b\}, \text{ because } \tilde{b}_0 < 0\\
&= (1/\tilde{b}_0)\ln[1-(\bigwedge\{f(x+h), h \in D_b\})/M]\\
&= (1/\tilde{b}_0)\ln[1-(\varepsilon_{D_b} f(x))/M].
\end{aligned}$$

Eq. 21 of the map of Asplund distances of f, $Asp^{\triangle\!\!\!\times}_{b_0} f$, is obtained from Eq. 18 and the previous expressions of $\lambda_{b_0} f$ and $\mu_{b_0} f$. □

Proof of proposition 4.
There is $\forall f, g \in \overline{\mathcal{I}}$, $\forall x \in D$,

$$\begin{aligned}
\lambda_b(f \vee g)(x) &= \bigvee \{(\widetilde{f \vee g}(x+h))/\tilde{b}(h), h \in D_b\} \\
&= \bigvee \{(\tilde{f}(x+h) \wedge \tilde{g}(x+h))/\tilde{b}(h), h \in D_b\}, \\
&= \bigvee \{(\tilde{f}(x+h)/\tilde{b}(h)) \vee (\tilde{g}(x+h)/\tilde{b}(h)), h \in D_b\}, \\
&= [\bigvee_{h \in D_b} \{\tilde{f}(x+h)/\tilde{b}(h)\}] \vee [\bigvee_{h \in D_b} \{\tilde{g}(x+h)/\tilde{b}(h)\}] \\
&= \lambda_b(f)(x) \vee \lambda_b(g)(x).
\end{aligned}$$

The second equality is obtained because $\tilde{f}$ is decreasing. The third equality is caused by $\tilde{b}(h) < 0$. According to definition 4.2, λ_b is a dilation. In addition,

$$\begin{aligned}
\lambda_b(O)(x) &= \lambda_b(f_0)(x) \\
&= \bigwedge_{h \in D_b} \{\alpha(x), \alpha(x) \geq (\tilde{0}(x+h)/\tilde{b}(h))\} \\
&= 0(x) = O(x).
\end{aligned}$$

Similarly, $\forall f, g \in \overline{\mathcal{I}}$, $\forall x \in D$

$$\begin{aligned}
\mu_b(f \wedge g)(x) &= \bigwedge \{(\widetilde{f \wedge g}(x+h)/\tilde{b}(h)), h \in D_b\} \\
&= \bigwedge \{(\tilde{f}(x+h) \vee \tilde{g}(x+h)/\tilde{b}(h)), h \in D_b\}, \\
&= [\bigwedge_{h \in D_b} \{\tilde{f}(x+h)/\tilde{b}(h)\}] \wedge [\bigwedge_{h \in D_b} \{\tilde{g}(x+h)/\tilde{b}(h)\}], \\
&= \mu_b(f)(x) \wedge \mu_b(g)(x).
\end{aligned}$$

According to definition 4.1, μ_b is an erosion. In addition,

$$\begin{aligned}
\mu_b(I)(x) &= \mu_b(f_M)(x) \\
&= \bigvee_{h \in D_b} \{\beta(x), \beta(x) \leq (\tilde{M}(x+h)/\tilde{b}(h))\} \\
&= +\infty(x) = I(x).
\end{aligned}$$

□

VERIFICATION OF THE METRIC PROPERTIES

Noyel and Jourlin (2017) have shown that the LIP-multiplicative Asplund distance $d_{asp}^{\triangle\!\!\!\!\times}$ is a metric in the space of equivalence classes $\mathcal{I}^{\triangle\!\!\!\!\times}$. In this section, we will demonstrate that the LIP-additive Asplund distance $d_{asp}^{\triangle\!\!\!\!+}$ is a metric in the space of equivalence classes $\mathcal{F}_M^{\triangle\!\!\!\!+}$, which represents the set of functions $h \in \mathcal{F}_M$ such that $h = f \triangle\!\!\!\!+ k$ for a constant k lying in $]-\infty, M[$.

Proof that the LIP-additive Asplund distance $d_{asp}^{\triangle\!\!\!\!+}$ is a metric in the space of equivalence classes $\mathcal{F}_M^{\triangle\!\!\!\!+}$.

Let $\mathcal{T} =]-\infty, M[$ be the space of real values less than M. In order to be a metric on $(\mathcal{F}_M^{\triangle\!\!\!\!+} \times \mathcal{F}_M^{\triangle\!\!\!\!+}) \to \mathbb{R}^+$, $d_{asp}^{\triangle\!\!\!\!+}$ must satisfy the four following properties:

- (Positivity): $\forall f^{\triangle\!\!\!\!+} \neq g^{\triangle\!\!\!\!+} \in \mathcal{F}_M^{\triangle\!\!\!\!+}$, $\forall x \in D$, as c_1 and c_2, can be expressed as $c_1 = \bigvee_{x \in D} \{f(x) \triangle\!\!\!\!- g(x)\}$ and $c_2 = \bigwedge_{x \in D} \{f(x) \triangle\!\!\!\!- g(x)\}$ (proof of Prop. 1), there is always $c_1 > c_2$.

$\Rightarrow d_{asp}^{\triangle\!\!\!\!+}(f, g) = c_1 \triangle\!\!\!\!- c_2 = (c_1 - c_2)/(1 - c_2/M) > 0$.

We have also demonstrated that the operator $\triangle\!\!\!\!-$ is strictly increasing.

- (Axiom of separation): Given the two equivalence classes $f^{\triangle\!\!\!\!+}, g^{\triangle\!\!\!\!+} \in \mathcal{F}_M^{\triangle\!\!\!\!+}$, we have the following implication:
$d_{asp}^{\triangle\!\!\!\!+}(f^{\triangle\!\!\!\!+}, g^{\triangle\!\!\!\!+}) = 0 \Rightarrow c_1 = c_2 = c$. In addition, according to definition 2, we have $c \triangle\!\!\!\!+ g \geq f \geq c \triangle\!\!\!\!+ g$. This implies that $c \triangle\!\!\!\!+ g = f$ and

$$f^{\triangle\!\!\!\!+} = g^{\triangle\!\!\!\!+}. \quad \text{(B.1)}$$

Reciprocally, there is $\forall f^{\triangle\!\!\!\!+}, g^{\triangle\!\!\!\!+} \in \mathcal{F}_M^{\triangle\!\!\!\!+}$, $(f^{\triangle\!\!\!\!+} = g^{\triangle\!\!\!\!+}) \Rightarrow (\exists k \in \mathcal{T},\ k \triangle\!\!\!\!+ g = f)$. In addition, according to definition 2, there is $c_1 = \inf\{c, f \leq c \triangle\!\!\!\!+ g\}$ and $c_2 = \sup\{c, c \triangle\!\!\!\!+ g \leq f\}$. This implies that $c_1 = c_2 = k$ and

$$d_{asp}^{\triangle\!\!\!\!+}(f, g) = 0. \quad \text{(B.2)}$$

Eq. B.1 and B.2 show that:
$\forall f^{\triangle\!\!\!\!+}, g^{\triangle\!\!\!\!+} \in \mathcal{F}_M^{\triangle\!\!\!\!+}$, $d_{asp}^{\triangle\!\!\!\!+}(f^{\triangle\!\!\!\!+}, g^{\triangle\!\!\!\!+}) = 0 \Leftrightarrow f^{\triangle\!\!\!\!+} = g^{\triangle\!\!\!\!+}$.

- (Triangle inequality): Let us define:
$d_{asp}^{\triangle\!\!\!\!+}(f^{\triangle\!\!\!\!+}, g^{\triangle\!\!\!\!+}) = c_1^a \triangle\!\!\!\!- c_2^a$, $d_{asp}^{\triangle\!\!\!\!+}(g^{\triangle\!\!\!\!+}, h^{\triangle\!\!\!\!+}) = c_1^b \triangle\!\!\!\!- c_2^b$ and $d_{asp}^{\triangle\!\!\!\!+}(f^{\triangle\!\!\!\!+}, h^{\triangle\!\!\!\!+}) = c_1^c \triangle\!\!\!\!- c_2^c$. Definition 2 gives the following system of equations:

$$\left.\begin{aligned}
c_1^a &= \inf\{c^a, f^{\triangle\!\!\!\!+} \leq c^a \triangle\!\!\!\!+ g^{\triangle\!\!\!\!+}\} \\
c_1^b &= \inf\{c^b, g^{\triangle\!\!\!\!+} \leq c^b \triangle\!\!\!\!+ h^{\triangle\!\!\!\!+}\} \\
c_1^c &= \inf\{c^c, f^{\triangle\!\!\!\!+} \leq c^c \triangle\!\!\!\!+ h^{\triangle\!\!\!\!+}\}
\end{aligned}\right\}.$$

This system implies that:

$$\begin{aligned}
& f^{\triangle\!\!\!\!+} \leq c_1^a \triangle\!\!\!\!+ g^{\triangle\!\!\!\!+} \leq c_1^a \triangle\!\!\!\!+ (c_1^b \triangle\!\!\!\!+ h^{\triangle\!\!\!\!+}) = (c_1^a \triangle\!\!\!\!+ c_1^b) \triangle\!\!\!\!+ h^{\triangle\!\!\!\!+} \\
& \Rightarrow c_1^c \leq c_1^a \triangle\!\!\!\!+ c_1^b \quad \text{(B.3)}
\end{aligned}$$

where the last inequality is obtained because c_1^c is the lowest value such that $f^{\triangle\!\!\!\!+} \leq c^c \triangle\!\!\!\!+ h^{\triangle\!\!\!\!+}$. Similarly, we have:

$$c_2^a \triangle\!\!\!\!+ c_2^b \leq c_2^c. \quad \text{(B.4)}$$

From Eq. B.3 and B.4, and knowing that $a ⨹ b = a + b - ab/M \le a + b$, we deduce that:

$$
\begin{aligned}
d^{⨹}_{asp}(f,h) &= c^c_1 ⨺ c^c_2\\
&\le (c^a_1 ⨹ c^b_1) ⨺ (c^a_2 ⨹ c^b_2) = (c^a_1 ⨺ c^a_2) ⨹ (c^b_1 ⨺ c^b_2)\\
&\le (c^a_1 ⨺ c^a_2) ⨹ (c^b_1 ⨺ c^b_2)\\
&\le (c^a_1 ⨺ c^a_2) + (c^b_1 ⨺ c^b_2) = d^{⨹}_{asp}(f,g) + d^{⨹}_{asp}(g,h).
\end{aligned}
$$

Finally, we have: $\forall f^{⨹}, g^{⨹}, h^{⨹} \in \mathcal{F}^{⨹}_M$, $d^{⨹}_{asp}(f^{⨹},h^{⨹}) \le d^{⨹}_{asp}(f^{⨹},g^{⨹}) + d^{⨹}_{asp}(g^{⨹},h^{⨹})$.

- (Axiom of symmetry): c_1 and c_2 can be expressed as: $c_1 = \bigvee_{x\in D}\{f(x) ⨺ g(x)\}$ and $c_2 = \bigwedge_{x\in D}\{f(x) ⨺ g(x)\}$. The Asplund metric becomes

$$
\begin{aligned}
d^{⨹}_{asp}(f,g) &= c_1 ⨺ c_2\\
&= \bigvee_{x\in D}\{f(x) ⨺ g(x)\} ⨺ \bigwedge_{x\in D}\{f(x) ⨺ g(x)\}\\
&= \bigvee_{x\in D}\{f(x) ⨺ g(x)\} ⨺ \bigwedge_{x\in D}\{⨺(g(x) ⨺ f(x))\}\\
&= \bigvee_{x\in D}\{f(x) ⨺ g(x)\} ⨹ \bigvee_{x\in D}\{g(x) ⨺ f(x)\}\\
&= \bigvee_{x\in D}\{g(x) ⨺ f(x)\} ⨹ \bigvee_{x\in D}\{f(x) ⨺ g(x)\}\\
&= d^{⨹}_{asp}(g,f)
\end{aligned}
$$

Therefore $\forall f^{⨹}, g^{⨹} \in \mathcal{F}^{⨹}_M$, $d^{⨹}_{asp}(f^{⨹},g^{⨹}) = d^{⨹}_{asp}(g^{⨹},f^{⨹})$.

□

INVARIANCES OF THE ROBUST TO NOISE METRICS

In this section, firstly, we will prove the invariance under LIP-multiplication by a scalar of the LIP-multiplicative Asplund metric with tolerance $d^{⨻}_{asp,p}$. Secondly, we will prove the invariance under LIP-addition of a constant of the LIP-additive Asplund metric with tolerance $d^{⨹}_{asp,p}$.

Proof of the invariance under LIP-multiplication by a scalar of the LIP-multiplicative Asplund metric with tolerance $d^{⨻}_{asp,p}$, property 6.

Given a real $\beta > 0$, according to definition 6, the metric: $d^{⨻}_{asp,p}(f, \beta ⨻ g)$ is equal to $\ln(\lambda'_\beta/\mu'_\beta)$. The factors λ'_β and μ'_β depend of the contrast function $\gamma^{⨻}_{(f,\beta⨻g)}$. Using Eq. 12 and 3, the contrast function $\gamma^{⨻}_{(f,\beta⨻g)}$ can be expressed as:

$$
\begin{aligned}
\gamma^{⨻}_{(f,\beta⨻g)} &= \frac{\ln(1-f/M)}{\ln(1-\beta ⨻ g/M)} = \frac{\ln(1-f/M)}{\ln(1-g/M)^\beta}\\
&= \frac{\ln(1-f/M)}{\beta\ln(1-g/M)} = (1/\beta)\gamma^{⨻}_{(f,g)}.
\end{aligned}
$$

The factor λ'_β is therefore equal to:

$$
\begin{aligned}
\lambda'_\beta &= \inf\{\alpha, \forall x, \gamma^{⨻}_{(f_{|D\setminus D'},\beta⨻g_{|D\setminus D'})}(x) \le \alpha\}\\
&= \inf\{\alpha, \forall x, (1/\beta)\gamma^{⨻}_{(f_{|D\setminus D'},g_{|D\setminus D'})}(x) \le \alpha\}\\
&= (1/\beta)\inf\{\alpha, \forall x, \gamma^{⨻}_{(f_{|D\setminus D'},g_{|D\setminus D'})}(x) \le \alpha\}\\
&= (1/\beta)\lambda'\\
&= \lambda'/\beta.
\end{aligned}
$$

Similarly, we have $\mu'_\beta = \mu'/\beta$.
The metric with tolerance $d^{⨻}_{asp,p}(f,\beta ⨻ g)$ becomes:
$d^{⨻}_{asp,p}(f,\beta ⨻ g) = \ln(\lambda'_\beta/\mu'_\beta) = \ln[(\lambda'/\beta)/(\mu'/\beta)] = \ln(\lambda'/\mu') = d^{⨻}_{asp,p}(f,g)$.
Similarly, we have $d^{⨻}_{asp,p}(\beta ⨻ f, g) = d^{⨻}_{asp,p}(f,g)$. □

Proof of the invariance under LIP-addition of a constant of the LIP-additive Asplund metric with tolerance $d^{⨹}_{asp,p}$, property 7.

Given $k \in]-\infty, M[$ and according to definition 10, the metric: $d^{⨹}_{asp,p}(f, k ⨹ g)$ is equal to $c'_{1,k} ⨺ c'_{2,k}$. The constants $c'_{1,k}$ and $c'_{2,k}$ depend of the contrast function $\gamma^{⨹}_{(f,k⨹g)}$. Using Eq. 30 and 2, the contrast function $\gamma^{⨹}_{(f,k⨹g)}$ can be expressed as:

$$
\gamma^{⨹}_{(f,k⨹g)} = f ⨺ (k ⨹ g) = (f ⨺ g) ⨺ k = \gamma^{⨹}_{(f,g)} ⨺ k.
$$

The factor $c'_{1,k}$ is therefore equal to:

$$
\begin{aligned}
c'_{1,k} &= \inf\{c, \forall x, \gamma^{⨹}_{(f_{|D\setminus D'},(g_{|D\setminus D'})⨹k)}(x) \le c\}\\
&= \inf\{c, \forall x, (\gamma^{⨹}_{(f_{|D\setminus D'},g_{|D\setminus D'})}(x) ⨺ k) \le c\}\\
&= \inf\{c, \forall x, \gamma^{⨹}_{(f_{|D\setminus D'},g_{|D\setminus D'})⨹k}(x) \le c\} ⨺ k\\
&= c'_1 ⨺ k.
\end{aligned}
$$

Similarly, we have $c'_{2,k} = c'_2 ⨺ k$.
The metric with tolerance $d^{⨹}_{asp,p}(f, k ⨹ g)$ becomes:
$d^{⨹}_{asp,p}(f, k ⨹ g) = c'_{1,k} ⨺ c'_{2,k} = (c'_1 ⨺ k) ⨺ (c'_2 ⨺ k) = c'_1 ⨺ c'_2 = d^{⨹}_{asp,p}(f,g)$.
Similarly, we have $d^{⨹}_{asp,p}(k ⨹ f, g) = d^{⨹}_{asp,p}(f,g)$. □

DETAILS OF THE ILLUSTRATION SECTION

Remark 1 (Segmentation details of Fig. 10)**.** *The h-minima have a height greater than the* 1.6^{th} *percentile of the map. Only the minima with an area less than the probe area and with a circular shape are kept.*

REFERENCES

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